GRAPH THEORY - EXAMPLE SHEET 2

Michaelmas 2021

Julian Sahasrabudhe

(1) For a graph G, show that $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$.

(2) Let G be a graph. Show that $e(G) \geqslant \binom{\chi(G)}{2}$.

(3) Show that every maximal planar graph on $n \geqslant 3$ vertices has $3n-6$ edges.

(4) Let G be a k-connected graph and let $y, x_1, \dots, x_k$ be distinct vertices in G. Show that there exists paths $P_1, \dots, P_k$, where P_i is a $y - x_i$ path and $P_1, \dots, P_k$ have no vertices in common, apart from the vertex y.

(5) An *independent set* in a graph $G = (V, E)$ is a subset $I \subset V$ so that $x \not\sim y$ for all $x, y \in I$. Let $G = (V, E)$ be a connected graph with $\Delta(G) \leqslant 3$ and $|V| \geqslant 10$. Show that there exists an independent set $I \subseteq V$ so that every odd cycle in G intersects I.

(6) Define the *edge chromatic number* of G, denoted $\chi'(G)$, to be the the chromatic number of the line graph of G. That is $\chi'(G) := \chi(L(G))$. Determine $\chi'(K_{n,n})$. Determine $\chi'(K_n)$.

(7) Say that a set $X \subseteq \mathbb{R}^2$ is *discrete* if every point $x \in \mathbb{R}^2$ has a neighborhood U so that $|U \cap X| \leqslant 1$. A *planar drawing* of an infinite graph is a drawing of G where the vertices of G form a discrete set and edges are represented by non-crossing polygonal arcs. Show that there is an infinite graph on a countable vertex set that cannot be drawn in the plane but has no subdivision of $K_{3,3}$ or K_5. (That is Kuratowski's theorem fails for infinite graphs).

(8) Let G be a bipartite graph with maximum degree Δ. Must we have $\chi'(G) = \Delta(G)$?

(9) Let $G = (V, E)$ be a graph where V, E are countably infinite. Show that $\chi(G) \leqslant k$ if and only if $\chi(H) \leqslant k$ for every finite subgraph H of G.

(10) Let $G = (V, E)$ be a k-connected graph, $k \geqslant 2$ and let $\{x_1, \dots, x_k\} \subseteq V$. Show that there exists a cycle containing each of the vertices $x_1, \dots, x_k$.

(11) For each $r \geqslant 2$ construct a graph G that does not contain a K_{r+1} and $\chi(G) > r$.

(12) A graph is *outer-planar* if it can be drawn in the plane so that all of its vertices are on the infinite face. Articulate a conjecture of the form "Let G be a graph with $|G| \geqslant 5$. G is outer-planar if and only if ....". Prove your conjecture.

(13) (*) Show there is a triangle free graph with chromatic number 2021.

(14) (*) Let $C \subseteq \mathbb{R}^2$ be a polygon with all vertices in $\mathbb{Z}^2$. Let $b(C)$ be the number of integer points on the boundary of C and let $i(C)$ be the number of integer points in the interior of C. Use Euler's formula to show that the area of C is $i(C) + b(C)/2 - 1$.