GRAPH THEORY - EXAMPLE SHEET 4

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(1) By considering the graph on $\mathbb{Z}_{17}$ in which x is joined to y if $x-y$ is a square modulo 17, show that $R(4)=18$.

(2) For $\ell, t \in \mathbb{N}$, define $R'_\ell(t)$ to be the smallest n so that every ℓ-colouring $c : E(K_n) \to \{1,\dots,\ell\}$ contains a monochromatic K_t. Show that $R'_\ell(t)$ exists.

(3) Let $\{I_i\}_{i\in\mathbb{N}}$ be a countable collection of closed, non-empty intervals in $\mathbb{R}$. Show that either there exists an infinite collection of pairwise disjoint intervals *or* there exists a $x \in \mathbb{R}$ that is covered by infinitely many intervals.

(4) Show that $R'_\ell(3) \leqslant \lceil e \cdot \ell! \rceil$, where $e = 2.718....$ is Euler's number.

(5) Calculate the eigenvalues of K_n and $K_{m,n}$.

(6) Let $G \sim G(n, p(n))$. Show that if $p(n)n^{2/3} \to 0$ then

$$\mathbb{P}(G \supset K_4) \to 0,$$

as $n \to \infty$ and if $p(n)n^{2/3} \to \infty$ then

$$\mathbb{P}(G \supset K_4) \to 1,$$

as $n \to \infty$. Here $p(n) = n^{2/3}$ is called the *threshold* for the event "$G \supset K_4$". What is the threshold for $G \supset K_r$?

(7) Let A be an infinite set of points in the plane, with no three points of A colinear. Prove that A contains an infinite set B such that no point of B is a convex combination of other points of B.

(8) Recall $\alpha(G)$ denotes the largest independent set in G. Show $\alpha(G(n, n^{-1/2})) \leqslant 4n^{1/2}\log n$, with probability tending to 1 as $n \to \infty$. Is it true that $\alpha(G(n, n^{-1/2})) \geqslant cn^{1/2}\log n$, with probability tending to 1, where $c > 0$ is some fixed constant?

(9) Prove that the matrix J (all of whose entries are 1) is a polynomial in the adjacency matrix of a graph G if and only if G is regular and connected.

(10) Let G be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique 4-cycle. Show that G is k-regular, for some k, and that the number of vertices of G is $k^2/2+1$. Show also that $k \in \{2, 4, 14, 22, 112, 994\}$.

(11) Let $G(1/2, \mathbb{N})$ be the random graph defined on vertex set $\mathbb{N}$ where every edge is included independently with probability 1/2. If two graphs G_1, G_2 are drawn independently from $G(1/2, \mathbb{N})$ show that G_1 is isomorphic to G_2 with probability 1.

(12) (*) Can the edges of K_{10} be decomposed into 3 disjoint copies of the Petersen graph?

(13) (*) Let A be an uncountable set and let $A^{(2)}$ be 2-coloured. Must there exist an uncountable monochromatic set in A?