GRAPH THEORY - EXAMPLE SHEET 1

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Julian Sahasrabudhe

(1) Carefully write down a proof of the following facts seen in lecture. a) an even cycle is bipartite and an odd cycle is not bipartite. b) If $x_1, \ldots, x_k$ is a closed walk in a graph G, with $x_{i-1} \neq x_{i+1}$ for $i = 2, ..., k-1$ and $x_{k-1} \neq x_2$ then $x_1, \ldots, x_k$ contains the vertices of a cycle in G.

(2) Show that every graph with at least two vertices has two vertices of the same degree.

(3) Show that in every connected graph there exists a vertex so that $G - v$ is connected.

(4) Show that if G is an acylic graph (a graph containing no cycle) then $e(G) \leqslant n - 1$.

(5) The *degree sequence* of a graph $G = (\{x_1, \ldots, x_n\}, E)$ is the sequence $d(x_1), \ldots, d(x_n)$.
For $n \geqslant 2$ let $1 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ be integers. Show that $(d_i)_{i=1}^n$ is a degree sequence of a tree if and only if $\sum_{i=1}^n d_i = 2n - 2$.

(6) Let $G = (V, E)$ be a graph. Show that there exists a partition of the vertex set $V = A \cup B$ so that

$$e(G[A]) + e(G[B]) \leqslant e(G)/2. \tag{1}$$

For $X \subseteq V$, $G[X]$ is defined as $G[X] = \big(X, \{ij \in E : i, j \in X\}\big)$. Show that we can find a partition S, T satisfying (1) and, additionally,

$$e(G[A]), e(G[B]) \leqslant e(G)/3.$$

(7) The *average degree* of a graph $G = (V, E)$ is $n^{-1} \sum_{x \in V} d(x)$. Show that if the average degree of G is d then G contains a subgraph with minimum degree $\geqslant d/2$.

(8) Say that a graph $G = (V, E)$ can be *decomposed* into cycles if E can be partitioned $E = E_1 \cup \ldots \cup E_k$, where each E_i is the edge set of a cycle. Show that G can be decomposed into cycles if and only if all degrees of G are even.

(9) The *clique number* of a graph G is the largest t so that G contains a complete graph on t vertices. Show that the possible clique numbers for a regular graph on n vertices are $1, 2, \ldots, \lfloor n/2 \rfloor$ and n.

(10) Let $G = (V, E)$ be a graph. Show that there is a partition $V = A \cup B$ so all the vertices in the graphs $G[A]$ and $G[B]$ are of even degree.

(11) An $n \times n$ Latin square (resp. $r \times n$ Latin rectangle) is an $n \times n$ (resp. $r \times n$) matrix, with each entry from $\{1, \ldots, n\}$, such that no two entries in the same row or column are the same. Prove that every $r \times n$ Latin rectangle may be extended to an $n \times n$ Latin square.

(12) (*) Let $G = (X \cup Y, E)$ be a countably infinite bipartite graph with the property that $|N(A)| \geqslant |A|$ for all $A \subseteq X$. Give an example to show that G need not contain a matching

saturating X. On the other hand, Show that if all of the degrees of G are finite then G *does* contain a matching saturating X. Does this remain true if G is uncountable?