GRAPH THEORY - EXAMPLE SHEET 4

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(1) By considering the graph on $\mathbb{Z}_{17}$ in which x is joined to y if $x - y$ is a square modulo 17, show that $R(4) = 18$.

(2) Prove an "infinite Ramsey theorem" for 3-sets: that is, show that for every red/blue colouring of $\mathbb{N}^{(3)}$ there exists an infinite set $X \subseteq \mathbb{N}$ so that $X^{(3)}$ is monochromatic.

(3) A Hamilton cycle in a tournament $T = (V, E)$, $|V| = n$, is a sequence $v_1, \dots, v_n$ where

$$(v_1, v_2), (v_2, v_3), \cdots, (v_{n-1}, v_n), (v_n, v_1) \in E.$$

Show that there exists tournaments with at least $n!/2^n$ Hamilton cycles. [Hint: consider a random tournament.]

(4) Define $R'_\ell(t)$ to be the smallest N so that every ℓ-colouring $c : E(K_N) \to [\ell]$ contains a monochromatic K_t. Show that $R'_\ell(t)$ exists.

(5) Show that $R'_\ell(3) \leqslant \lceil e \cdot \ell! \rceil$, where $e = 2.718....$ is Euler's number.

(6) Complete "Step 1" of the Erdős–Stone theorem (as seen in lecture). Let $\alpha \in (0, 1)$, $\varepsilon > 0$ and G be a graph with $|G| > 10/\varepsilon$ and $e(G) \geqslant (\alpha + \varepsilon)|G|^2$ then there exists a subgraph G' of G with $|G'| > \varepsilon^{1/2}|G|$ and $\delta(G) \geqslant \alpha|G'|$.

(7) Complete "Step 2" of the Erdős–Stone theorem (as seen in lecture): For $\alpha \in (0, 1)$, let $G = (A \cup B, E)$ be a bipartite graph so that

$$\frac{1}{|B|} \sum_{v \in B} d(v) \geqslant (\alpha + \varepsilon)|A|.$$

Show there is $B' \subseteq B$ with $|B'| \geqslant \varepsilon|B|$ so that $d(v) \geqslant \alpha|A|$, for all $v \in B'$. Also take a moment to sketch how this and the previous exercise fit into the proof of the Erdős–Stone theorem.

(8) Calculate the eigenvalues of K_n and $K_{m,n}$.

(9) Let A be an infinite set of points in the plane, with no three points of A colinear. Prove that A contains an infinite set B such that no point of B is a convex combination of other points of B.

(10) Recall that an independent set in a graph is a set of vertices with no two adjacent, let $\alpha(G)$ be the largest independent set in G. Show $\alpha(G(n, n^{-1/2})) \leqslant 4n^{1/2} \log n$, with probability tending to 1 as $n \to \infty$. Is this bound sharp?

(11) Prove that the matrix J (all of whose entries are 1) is a polynomial in the adjacency matrix of a graph G if and only if G is regular and connected.

(12) Let $G(1/2, \mathbb{N})$ be the random graph defined on vertex set $\mathbb{N}$ where every edge is included independently with probability 1/2. If two graphs G_1, G_2 are drawn independently from $G(1/2, \mathbb{N})$ show that G_1 is isomorphic to G_2 with probability 1.

(13) (*) Can the edges of K_{10} be decomposed into 3 disjoint copies of the Petersen graph?

(14) (*) Let A be an uncountable set and let $A^{(2)}$ be 2-coloured. Must there exist an uncountable monochromatic set in A?