GRAPH THEORY - EXAMPLE SHEET 3

Lent 2021

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(1) Show that Dirac's theorem is sharp: Show that for all odd n there exists a graph on G on n vertices with $\delta(G) = (n-1)/2$ and no Hamilton cycle.

(2) Let G be a graph. Show that $e(G) \geqslant \binom{\chi(G)}{2}$.

(3) Given n distinct points $x_1, \ldots, x_n \in \mathbb{R}^2$ and n distinct lines $\ell_1, \ldots, \ell_n \subseteq \mathbb{R}^2$, show that there are at most $10n^{3/2}$ point-line incidences, that is

$$|\{(i,j) : x_i \in \ell_j\}| < 10n^{3/2}.$$

(4) Determine $\chi'(K_{n,n})$. Determine $\chi'(K_n)$.

(5) Show every graph with $\delta(G) \geqslant 3$ contains a subdivision of K_4.

(6) An *independent set* in a graph is a collection of vertices where no two are adjacent. Show that if $G = (V, E)$ is a graph then there is an independent set $I \subseteq V$ with

$$|I| \geqslant \sum_{x \in V} \frac{1}{d(x)+1}.$$

[Hint: consider a random ordering of V]

(7) Let G be a bipartite graph with maximum degree Δ. Must we have $\chi'(G) = \Delta(G)$?

(8) A *Tournament* is a complete graph where every edge $\{u, v\}$ is given a unique direction (u, v) or (v, u). So the (directed) edges of a tournament on $\{1, \ldots, n\}$ are given by a set of ordered pairs E where exactly one of (i, j) or $(j, i) \in E$, for all $i < j \in [n]$. Show that a tournament contains a path of length $n-1$: a sequence of vertices $x_1 x_2, \ldots, x_n$ so that the directed edge $(x_i, x_{i+1}) \in E$ for all $1 \leqslant i \leqslant n-1$.

(9) In a tournament T, a *king* is a vertex v which can reach any other point via a directed path of length at most 2. I.e. for all $y \in T - v$, either $(v, y) \in E$ or there exists x so that $(v, x), (x, y) \in E$. Show that every tournament has a king.

(10) Let G be a countable graph. Show that $\chi(G) \leqslant k$ if and only if $\chi(H) \leqslant k$ for every finite subgraph H of G.

(11) Can $K_{4,4}$ be drawn in the torus? What about $K_{5,5}$?

(12) For each $r \geqslant 2$ construct a graph that does not contain a K_{r+1} but is not r-partite.

(13) A bow-tie is the graph on 5 vertices consisting of two triangles which share a single vertex. Show that if $|G| > 5$ and $e(G) > \frac{n^2}{4} + 1$ then G contains a bow-tie.

(14) Let G be a 3-regular graph and let $e \in E$. Show that G has an even number of Hamilton cycles through e (Recall 0 is an even number).

(15) (*) Show there is a triangle free graph with chromatic number 2021.