GRAPH THEORY - EXAMPLE SHEET 2

January 2021

Julian Sahasrabudhe

(1) Show that the Petersen graph is not planar in two different ways.
(2) Show that every maximal planar graph on $n \geqslant 3$ vertices has $3n-6$ edges.
(3) For a graph G, show that $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$.
(4) Let G be a k-connected graph and let $y, x_1, \ldots, x_k$ be distinct vertices in G. Show that there exists paths $P_1, \ldots, P_k$, where P_i is a $y - x_i$ path and $P_1, \ldots, P_k$ have no vertices in common, apart from the vertex y.
(5) Say that a set $X \subseteq \mathbb{R}^2$ is *discrete* if every point $x \in R^2$ has a neighborhood U so that $|U \cap X| \leqslant 1$. A *planar drawing* of an infinite graph is a drawing of G where the vertices of G form a discrete set and edges are represented by non-crossing polygonal arcs. Show that there is an infinite graph on a countable vertex set that cannot be drawn in the plane but has no subdivision of $K_{3,3}$ or K_5. (That is Kuratowski's theorem fails for infinite graphs).
(6) Let $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the sphere in three dimensions and let $C_1, \ldots, C_k$ be great circles on S that don't all meet at a common point. Show that there exists a point that meets exactly two of the great circles $C_1, \ldots, C_k$.
(7) Let G be a plane graph suppose that each vertex of G is coloured red or blue. Show that there is a vertex v so that if one considers the neighbours of v in clockwise order, then the colour changes at most twice.
(8) Let $X \subseteq \mathbb{R}^2$ be a finite set of points not all on a line and suppose that each point of X is coloured either red or blue. Show that there exists a line ℓ so that $|\ell \cap X| \geqslant 2$ and all of the points in $\ell \cap X$ are the same colour.
(9) A polyhedron is a set $P \subseteq \mathbb{R}^3$ defined by the intersection of finitely many half spaces. Recall a half space is a set of the form $\{x \in \mathbb{R}^3 : \langle x, v \rangle < c\}$). If P if a bounded polyhedron in $\mathbb{R}^3$ the *skeleton* of P is a graph where the vertices are the vertices of the polyhedron (where at least three faces meet) and the edges are edges of the polyhedron (where two faces meet). Show that the skeleton of a bounded polyhedron in $\mathbb{R}^3$ is planar.
(10) A platonic solid is a polyhedron in $\mathbb{R}^3$ where every face has the same number of edges on its boundary and every vertex is incident with the same number of edges. Show that that there are only 5 platonic solids: the cube, the tetrahedron, the octahedron, dodecahedron and the icosohedron.
(11) Let $G = (V,E)$ be a k-connected graph, $k \geqslant 2$ and let $\{x_1, \ldots, x_k\} \subseteq E$. Show that there exists a cycle containing each of the vertices $x_1, \ldots, x_k$.
(12) A graph is *outer-planar* if it can be drawn in the plane so that all of its faces are on the infinite face. Articulate a conjecture of the form "Let G be a graph with $|G| \geqslant 5$. G is outer-planar if and only if ....". Prove your conjecture.
(13) Show that every planar graph has a drawing in the plane where each line segment (representing an edge) is a straight line.
(14) (*) Let $C \subseteq \mathbb{R}^2$ be a polygon with all vertices in $\mathbb{Z}^2$. Let $b(C)$ be the number of integer points on the boundary of C and let $i(C)$ be the number of integer points in the interior of C. Use Euler's formula to show that the area of C is $i(C) + b(C)/2 - 1$.