Graph Theory (2019–20) Example Sheet 3 of 4

1. Let $G = K_{s,t}$. Show that A has eigenvalues $\pm\sqrt{st}$ (once each) and 0 ($s+t-2$ times), and L has eigenvalues 0 and $s+t$ (once each), s ($t-1$ times) and t ($s-1$ times).

2. Prove that the matrix J (all of whose entries are 1) is a polynomial in the adjacency matrix of a graph G if and only if G is regular and connected.

3. Let G be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique 4-circuit. Show that $|G| \in \{3, 9, 99, 243, 6273, 494019\}$.

4. Show that $e(G) \geq \binom{\chi(G)}{2}$ holds for every graph G.

5. Prove that a triangulated plane map is 3-colourable unless it is K_4.

6. Let $p_G(x) = \sum_{i=0}^{|G|}(-1)^i a_i x^{|G|-i}$ be the chromatic polynomial of G. Find $p_G(x)$ when $G = K_{3,m}$ and when G is the wheel W_k (a circuit C_k plus a vertex joined to all of it).

Prove that $a_i \geq 0$, $a_0 = 1$, $a_1 = e(G)$ and $a_2 = \binom{e(G)}{2} - t(G)$, where $t(G)$ is the number of triangles in G.

7. Find graphs G and H with $|G| = |H|$, $e(G) = e(H)$ and $\chi(G) > \chi(H)$, such that there are more ways to colour G than H when the number of available colours is large.

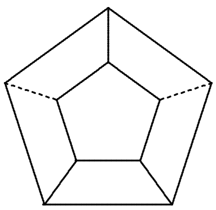
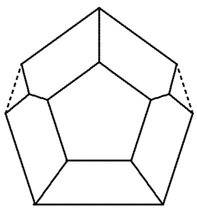
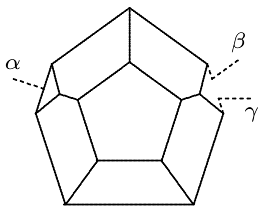
8. What is $\chi'(K_{m,n})$? What is $\chi'(K_n)$?

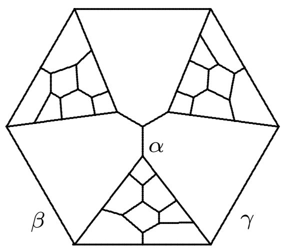
9. Let G be a graph of order n with complement $\overline{G}$. Show that $2\sqrt{n} \leq \chi(G)+\chi(\overline{G}) \leq n+1$.

10. Let $n = 2^p$. Show that K_{n+1} is not the union of p bipartite graphs but that K_n is. Deduce that among any $2^p + 1$ points in the plane there are three that determine an angle of size at least $\pi(1-(1/p))$.

11. Show that an Eulerian plane map is 2-colourable.

12. Verify Tutte's counterexample (on the right) to Tait's conjecture (maybe the dashes help).



13. Show that $\max\{\chi(G) : G \text{ embeds on the projective plane}\} = 6$.

14. Prove that, if G is countable and $\chi(H) \leq k$ for every finite $H \subset G$, then $\chi(G) \leq k$.

$^+$**15.** You are at a party where no-one has more friends than you do. You discover that every two people there have exactly one mutual friend present. Prove that everybody is your friend.

$^+$**16.** Construct a triangle-free graph of chromatic number 1526.