Graph Theory (2019–20) Example Sheet 2 of 4

1. For which n and m is $K_{n,m}$ Hamiltonian? Is the Petersen graph (on sheet 1) Hamiltonian?

2. A *tournament* is a complete graph in which each edge uv is given a direction, either from u to v or from v to u. Show that a tournament must contain a Hamiltonian path, i.e. a directed path through all the vertices. Must it contain a Hamiltonian circuit?

3. Construct a graph of order n with no Hamiltonian circuit and with size $\binom{n}{2} - (n-2)$. Show that no greater size can be achieved.

4. The points $a_1, \ldots, a_n$ lie in the plane, with $||a_i - a_j|| \le 1$, $1 \le i < j \le n$. Prove that at most $n^2/3$ of these distances exceed $1/\sqrt{2}$.

5. Let G have $n \ge r+2 \ge 4$ vertices and $t_r(n)+1$ edges. Prove that, for every p in the range $r+1 \le p \le n$, G has a subgraph of order p and size at least $t_r(p)+1$. Infer that G contains all but one edge of a K_{r+2}.

6. Show that every graph of order $n \ge 6$ and size $\lfloor n^2/4 \rfloor + 1$ contains a C_5.

7. Prove that for $n \ge 5$ every graph of order n with $\lfloor n^2/4 \rfloor + 2$ edges contains two triangles with exactly one vertex in common.

8. For each $r \ge 2$, construct a graph that does not contain K_{r+1} but that is not r-partite.

9. Find $\lim_{n\to\infty} \mathrm{ex}(n; P)/\binom{n}{2}$, where P is the Petersen graph.

10. The *upper density* ud(G) of an infinite graph G is the supremum of the densities of its large finite subgraphs; that is,

$$\mathrm{ud}(G) = \lim_{n\to\infty} \sup \{\, e(H)/\tbinom{|H|}{2} : H \subset G,\, n \le |H| < \infty \} .$$

Show that, for every G, $\mathrm{ud}(G) \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1 - \frac{1}{r}, \ldots, 1\}$.

11. Prove that if $|G| = n$ and $e(G) > \dfrac{n}{4}\{1 + \sqrt{4n-3}\}$ then G contains C_4. More generally, show that if $e(G) > \dfrac{n}{4}\{1 + \sqrt{1 + 4(n-1)(t-1)}\}$ then G contains $K_{2,t}$.

12. Given a graph G with vertex set $[n]$, let $s(G)$ be the smallest size of a set S having subsets $S_1, \ldots, S_n \subset S$ with $S_i \cap S_j \ne \emptyset$ iff $ij \in E(G)$. Show that $s(G) \le e(G)$, with equality iff G has no triangles. Prove that $s(G) \le n^2/4$ for all G.

13. Show that the maximum size of a graph of order n having only even circuits is $\lfloor n^2/4 \rfloor$. Show that the maximum size of a graph of order n having only odd circuits is $\lfloor 3(n-1)/2 \rfloor$.

$^{+}$**14.** Show that an r-regular graph of order $2r+1$ is Hamiltonian.