Graph Theory (2019–20) Example Sheet 1 of 4

1. Show that every graph (of order at least two) has two vertices of the same degree.

2. Show that every connected graph G has a vertex v such that $G - v$ is connected.

3. The *complement* of the graph $G = (V, E)$ is $\overline{G} = (V, V^{(2)} - E)$. A graph isomorphic to its complement is *self-complementary*. Show that there is a self-complementary graph of order n if and only if $n \equiv 0$ or $1 \pmod 4$.

4. Let $(d_i)_1^n$ be a sequence of integers. Show that there is a tree with degree sequence $(d_i)_1^n$ if and only if $d_i \geq 1$ for all i and $\sum_{i=1}^n d_i = 2n - 2$.

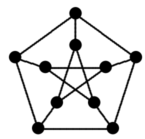
5. Let $T_1, \dots, T_k$ be subtrees of a tree T, any two of which have at least one vertex in common. Prove that there is a vertex in all the T_i.

6. Let G be a graph. Show that its vertex set V has a partition $V = V_1 \cup V_2$ such that

$$e(G[V_1]) + e(G[V_2]) \leq \frac{1}{2} e(G).$$

Show also that one may also demand that each V_i span at most a third of the edges; that is, $e(G[V_i]) \leq \frac{1}{3} e(G), \quad i = 1, 2.$

7. Draw the maps of the five Platonic solids. What are the dual maps?



8. Give two distinct arguments for why the *Petersen* graph (shown) is non-planar.

9. Show that every maximal planar graph of order $n \geq 3$ has $3n - 6$ edges.

10. Prove that every planar graph has a drawing in the plane in which every edge is a straight line segment.

11. Let G be a bipartite graph with bipartition X, Y having a matching from X into Y. Prove that there is a vertex $x \in X$ such that, for every edge xy, there is a matching from X to Y that contains xy.

12. An $n \times n$ Latin square (resp. $r \times n$ Latin rectangle) is an $n \times n$ (resp. $r \times n$) matrix, with each entry from $\{1, \dots, n\}$, such that no two entries in the same row or column are the same. Prove that every $r \times n$ Latin rectangle may be extended to an $n \times n$ Latin square.

13. Must $\kappa(G - v) \leq \kappa(G)$ hold for all $v \in G$? Show that $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

14. Prove that a graph G is k-connected iff $|G| \geq k+1$ and for any $U \subset V(G)$ with $|U| \geq k$ and for any vertex $x \notin U$, there is an x–U *fan*, that is, k paths from x to U, any pair of paths having only the vertex x in common.

15. Prove that if G is k-connected $(k \geq 2)$ and $\{x_1, x_2, \dots, x_k\} \subset V(G)$ then there is a cycle in G of length at least $k + 1$ that contains all x_i, $1 \leq i \leq k$.

$^+$**16.** Each of n ageing dons has an item of gossip to impart. News is passed on by telephone: when two dons communicate, they share all the scandal they have gleaned thus far. How many calls are needed before each don knows all?