Mich. 2018 **GRAPH THEORY—EXAMPLES 4** PAR

1, Let $p \in (0,1)$ be constant. Show that a. e. $G \in \mathcal{G}(n,p)$ has diameter 2.

2. Find (with justification) a real number α such that $n \mapsto 1/n^\alpha$ is a sharp threshold for $G \in \mathcal{G}(n,p)$ to contain a K_4.

3. A *tournament* of order n is a function $f\colon E(K_n) \to V(K_n)$ such that $f(e) \in e$ for all $e \in E(K_n)$. If $f(uv) = u$, we say that u *beats* v. Show that, for every positive integer k, there is a tournament of some order n such that for any vertices $v_1, v_2, \ldots, v_k \in K_n$ there is some $u \in K_n$ that beats all of them.

4. Let G be a (not necessarily planar) graph with $|G| = n$ and $e(G) = m$. Suppose that G is drawn in the plane, but with edges allowed to cross. Let t be the number of pairs of edges which cross. Show that $t \geqslant m - 3n + 6$.

Suppose now $m \geqslant 4n$. By considering a random set $W \subset V(G)$ containing each vertex of G independently with probability $4n/m$, show that in fact $t \geqslant m^3/64n^2$.

5. Let $p = \lambda \log n/n$ where $\lambda > 0$ is constant. Show that if $\lambda < 1$ then a. e. $G \in \mathcal{G}(n,p)$ has an *isolated vertex,* i. e. a vertex of degree zero, whereas if $\lambda > 1$ then a. e. $G \in \mathcal{G}(n,p)$ has no isolated vertex.

6 Show that $R(s,t) \geqslant n - \binom{n}{s}p^{\binom{s}{2}} - \binom{n}{t}(1-p)^{\binom{t}{2}}$ for all $n \in \mathbb{N}$ and $p \in (0,1)$. By choosing p and n appropriately, deduce that $R(4,t) = \Omega\big((t/\log t)^{\frac{3}{2}}\big)$.

7. Find the chromatic polynomial of the n-cycle.

8. Let G be a graph with chromatic polynomial

$$p(X) = X^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_0.$$

Show that the coefficients a_i alternate in sign. Show also that if G has m edges and t triangles then $a_{n-2} = \binom{m}{2} - t$.

9. Find the eigenvalues of the complete r-partite graph $K_r(t)$ and of the cycle C_n.

10. Let G be a connected graph of maximal degree Δ.

(a) Show that if $-\Delta$ is an eigenvalue of G then G is bipartite.

(b) Show that if G is bipartite and μ is an eigenvalue of G then $-\mu$ is also an eigenvalue of G, and that μ and $-\mu$ have the same multiplicity.

11. Let G be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique 4-cycle. Show that $|G| \in \{3, 9, 99, 243, 6273, 494019\}$.

12. Any two members of a certain College have a unique common enemy who is also a member of the College. Show that there is some College member (the 'Junior Bursar') who is everyone else's enemy.