Mich. 2018 **GRAPH THEORY—EXAMPLES 3** PAR

1(a) For which m and n is $K_{m,n}$ planar?

(b) For each $n \geqslant 4$, let G_n be the graph with vertex set $[n]$, and ij an edge iff $i - j \equiv \pm 1$ or $\pm 2 \pmod{n}$. For which n is G_n planar?

2. Show, without assuming the four-colour theorem, that every triangle-free planar graph is four-colourable.

3. Let (G_n) be a sequence of graphs with $|G_n| = n$ for each n. If there is some $\varepsilon > 0$ such that we have $e(G_n) \geq (\frac{2}{3} + \varepsilon)\binom{n}{2}$ for every n, why must every planar graph be a subgraph of some G_n? Show that this need not be the case if instead $e(G_n)/\binom{n}{2} \to 2/3$.

4. Where is the error in the 'proof' of the Four-Colour Theorem given in lectures (summarized overleaf)?

5. Suppose G is a minimal non-4-colourable plane triangulation. Without assuming the Four Colour Theorem:

(a) show that G does not contain the Birkhoff diamond;

(b) by counting faces, show that G must contain a vertex of degree 5 with two neighbours each of degree 5 or 6; and

(c) by applying the discharging rule that each vertex of degree 5 gives charge $\frac{1}{3}$ to each of its neighbours of degree at least 7, show that G must contain a vertex of degree 5 with *either* a neighbour of degree 5 *or* two consecutive neighbours of degree 6.

6. What is $\chi'(K_{n,n})$? What is $\chi'(K_n)$?

7. Show that every Δ-regular bipartite graph is Δ-edge-colourable. Is it true that for every bipartite graph G we have $\chi'(G) = \Delta(G)$?

8. Let G be a graph and $v \in G$. Must $\kappa(G - v) \leqslant \kappa(G)$?

9. Show that, for any graph G, $\kappa(G) \leqslant \lambda(G)$, and that if G is 3-regular then $\kappa(G) = \lambda(G)$. Given positive integers $k \leqslant \ell$, construct a graph G with $\kappa(G) = k$ and $\lambda(G) = \ell$.

10. Let G be a bipartite graph with vertex classes X and Y. Show that if G has a matching from X to Y then there exists $x \in X$ such that every edge containing x extends to a matching from X to Y.

11. An $n \times n$ *Latin square* (resp. $r \times n$ *Latin rectangle*) is an $n \times n$ (resp. $r \times n$) matrix, with entries in $[n]$, such that no two entries in the same row or column are the same. Prove that every $r \times n$ Latin rectangle may be extended to an $n \times n$ Latin square.

12. Let G be a k-connected graph. Suppose that $v \in G$ and $U \subset V(G) - \{v\}$ with $|U| \geq k$. Show that G contains k vU-paths any two of which have only the vertex v in common.

13. Let G be a k-connected graph ($k \geqslant 2$), and let $x_1, x_2, \ldots, x_k$ be vertices of G. Show that there is a cycle in G containing all the x_i.

14. Let G be an infinite bipartite graph with vertex classes X and Y such that $|\Gamma(A)| \geq |A|$ for every $A \subset X$. Give an example to show that G need not contain a matching from X to Y . Show however that if G is countable and every vertex in X has finite degree then G does contain a matching from X to Y. $^+$Does this remain true if G is uncountable?

PROOF OF THE FOUR-COLOUR THEOREM

Let G be a planar graph. We shall prove that G is 4-colourable.

We proceed by induction on $|G|$. If $|G| \leqslant 4$ then the result is trivial, so suppose $|G| = n > 4$. Choose $v \in G$ of minimal degree and let $H = G - v$. By the induction hypothesis, we have a 4-colouring c of H. As in the proof of the Five-Colour Theorem (5CT), $d(v) \leqslant 5$. Draw G. If some colour is missing on $\Gamma(v)$ we can use that colour at v, so assume not. There are three cases to consider.

(i) $d(v) = 4$**, and** v **has neighbours** x_1**,** x_2**,** x_3**,** x_4 **in clockwise order with** $c(x_i) = i$**.** There cannot be both a 13-path from x_1 to x_3 and a 24-path from x_2 to x_4, so as in the proof of 5CT we can make some colour swap and colour v.

(ii) $d(v) = 5$ **and** v **has neighbours** x_1**,** x_1'**,** x_2**,** x_3**,** x_4 **in clockwise order with** $c(x_i) = i$ **and** $c(x_1') = 1$**.** There must be a 24-path from x_2 to x_4 or we can make some colour swap and colour v. But then there can be a 13-path from x_3 to neither x_1 nor x_1'. So swap colours 1 and 3 on the 13-component of x_3 and give v colour 3.

(iii) $d(v) = 5$ **and** v **has neighbours** x_1**,** x_2**,** x_1'**,** x_3**,** x_4 **in clockwise order with** $c(x_i) = i$ **and** $c(x_1') = 1$**.** There must be a 23-path from x_2 to x_3 and a 24-path from x_2 to x_4 or we can make some colour swap and colour v. But then there can be neither a 13-path from x_1 to x_3 nor a 14-path from x_1' to x_4. So swap colours 1 and 3 on the 13-component of x_1; swap colours 1 and 4 on the 14-component of x_1'; and give v colour 1.