Mich. 2018 **GRAPH THEORY—EXAMPLES 2** PAR

1. Let $P = (\{0,1,2,3,4,5,6,7,8,9\}, \{12,23,34,45,51,16,27,38,49,50,68,80,07,79,96\})$. What is $\mathrm{ex}\,(P)$? *(P is called the Petersen graph.)*

2. For each $r \geqslant 3$, construct a graph of chromatic number r that contains no K_r.

3. Let G be a graph. Define a relation $\to$ on $V(G)$ by $u \to v$ iff there is a path in G from u to v. Show that $\to$ is an equivalence relation. Hence deduce that G can be written uniquely as a disjoint union of connected subgraphs each with at least one vertex.

4. Let G be a graph of order n. Suppose that $d(x)+d(y) \geqslant n$ for all distinct, non-adjacent x, $y \in G$. Show that G is Hamiltonian. Deduce that any graph of order $n \geqslant 3$ with $\binom{n}{2} - n + 3$ edges must be Hamiltonian. For each $n \geqslant 3$, give an example of a graph with $\binom{n}{2} - n + 2$ edges which is not Hamiltonian.

5. Let G be a triangle-free graph with $|G| = n$ and $e(G) = m$. Show that if $xy \in E(G)$ then $d(x) + d(y) \leqslant n$. Hence show that $\sum_{v \in G} d(v)^2 \leqslant nm$ and deduce Mantel's Theorem.

6. Show that $\mathrm{ex}\,(n; K_{2,t}) \leqslant \frac{n}{4}\left(1 + \sqrt{1 + 4(n-1)(t-1)}\right)$.

7. What is $\mathrm{ex}\,(n; K_{1,t})$?

8. Let G be a graph of order $n \geqslant 5$ with $e(G) \geqslant \lfloor \frac{n^2}{4} \rfloor + 2$. Show that G must contain two triangles with precisely one vertex in common.

9. Let x_1, x_2, $\ldots$, x_{3n} be points in the plane such that no two of them are more than distance 1 apart. Prove that at most $3n^2$ of the distances $\|x_i - x_j\|$ $(i < j)$ are greater than $1/\sqrt{2}$.

10. Without assuming the Erdős-Stone theorem, show that $\mathrm{ud}\,(G)$ is well-defined for all infinite graphs G. That is, show that $\lim_{n\to\infty} \sup\{D(H) : H \subset G, |H| = n\}$ exists.

11. For each positive integer n, let g_n be the largest integer k such that it is possible to colour k edges of the complete graph K_n blue or yellow without creating a monochromatic (blue or yellow) triangle. Show that $g_n/\binom{n}{2}$ converges, and find $\lim_{n\to\infty} g_n/\binom{n}{2}$.

12. How many edges can a graph G of order n containing precisely one triangle have?

13. Let $x_1, x_2, \ldots \in \mathbb{R}^d$ be distinct and, for each n, let $f(n)$ be the number of pairs of points from x_1, $\ldots$, x_n that are at distance 1: $f(n) = |\{(i,j) : 1 \leqslant i < j \leqslant n, \|x_i - x_j\| = 1\}|$. Show that if $d = 3$ then $f(n)/\binom{n}{2} \to 0$ as $n \to \infty$. Show more generally that if $d \geqslant 3$ then $f(n) \leqslant (1 - \frac{1}{D} + o(1))\binom{n}{2}$, where $D = \lfloor d/2 \rfloor$.

14. Show that an r-regular graph of order $2r+1$ must be Hamiltonian.

$^{+}$15. Construct a triangle-free graph of chromatic number 2018.