Mich. 2017 **GRAPH THEORY—EXAMPLES 3** PAR

1. For which m and n is $K_{m,n}$ planar?

2. For each $n \geqslant 4$, let G_n be the graph with vertex set $[n]$, and ij an edge iff $i - j \equiv \pm 1$ or $\pm 2 \pmod{n}$. For which n is G_n planar?

3. Show, without assuming the four-colour theorem, that every triangle-free planar graph is four-colourable.

4. Let (G_n) be a sequence of graphs with $|G_n| = n$ for each n. If there is some $\varepsilon > 0$ such that we have $e(G_n) \geq (\frac{2}{3} + \varepsilon)\binom{n}{2}$ for every n, why must every planar graph be a subgraph of some G_n? Show that this need not be the case if instead $e(G_n)/\binom{n}{2} \to 2/3$.

5. What is $\chi'(K_{n,n})$? What is $\chi'(K_n)$?

6. Show that every Δ-regular bipartite graph is Δ-edge-colourable. Is it true that for every bipartite graph G we have $\chi'(G) = \Delta(G)$?

7. Let G be a graph and $v \in G$. Must $\kappa(G - v) \leqslant \kappa(G)$?

8. Show that, for any graph G, $\kappa(G) \leqslant \lambda(G)$, and that if G is 3-regular then $\kappa(G) = \lambda(G)$. Given positive integers $k \leqslant \ell$, construct a graph G with $\kappa(G) = k$ and $\lambda(G) = \ell$.

9. Let G be a bipartite graph with vertex classes X and Y. Show that if G has a matching from X to Y then there exists $x \in X$ such that every edge containing x extends to a matching from X to Y.

10. An $n \times n$ *Latin square* (resp. $r \times n$ *Latin rectangle*) is an $n \times n$ (resp. $r \times n$) matrix, with entries in $[n]$, such that no two entries in the same row or column are the same. Prove that every $r \times n$ Latin rectangle may be extended to an $n \times n$ Latin square.

11. Let G be a k-connected graph. Suppose that $v \in G$ and $U \subset V(G) - \{v\}$ with $|U| \geq k$. Show that G contains k vU-paths any two of which have only the vertex v in common.

12. Let G be a k-connected graph ($k \geqslant 2$), and let $x_1, x_2, \ldots, x_k$ be vertices of G. Show that there is a cycle in G containing all the x_i.

13. Let G be an infinite bipartite graph with vertex classes X and Y such that $|\Gamma(A)| \geq |A|$ for every $A \subset X$. Give an example to show that G need not contain a matching from X to Y . Show however that if G is countable and every vertex in X has finite degree then G does contain a matching from X to Y. $^+$Does this remain true if G is uncountable?