

1. Let $p \in (0, 1)$ be constant. Show that a. e. $G \in \mathcal{G}(n, p)$ has diameter 2.
2. Find (with justification) a real number *alpha* such that $p = 1/n^\alpha$ is a threshold for $G \in \mathcal{G}(n, p)$ to contain a K_4 in the sense that if $pn^\alpha \rightarrow 0$ then a. e. $G \in \mathcal{G}(n, p)$ contains no K_4 , while if $pn^\alpha \rightarrow \infty$ then a. e. $G \in \mathcal{G}(n, p)$ does contain a K_4 .
3. A *tournament* of order n is a function $f: E(K_n) \rightarrow V(K_n)$ such that $f(e) \in e$ for all $e \in E(K_n)$. If $f(uv) = u$, we say that u *beats* v . Show that, for every positive integer k , there is a tournament of some order n such that for any vertices $v_1, v_2, \dots, v_k \in K_n$ there is some $u \in K_n$ that beats all of them.
4. Let G be a (not necessarily planar) graph with $|G| = n$ and $e(G) = m$. Suppose that G is drawn in the plane, but with edges allowed to cross. Let t be the number of pairs of edges which cross. Show that $t \geq m - 3n + 6$.
Suppose now $m \geq 4n$. By considering a random set $W \subset V(G)$ containing each vertex of G independently with probability $4n/m$, show that in fact $t \geq m^3/64n^2$.
5. Let $p = \lambda \log n/n$ where $\lambda > 0$ is constant. Show that if $\lambda < 1$ then a. e. $G \in \mathcal{G}(n, p)$ has an *isolated vertex*, i. e. a vertex of degree zero, whereas if $\lambda > 1$ then a. e. $G \in \mathcal{G}(n, p)$ has no isolated vertex.
- 6 Show that $R(s, t) \geq n - \binom{n}{s}p^{\binom{s}{2}} - \binom{n}{t}(1-p)^{\binom{t}{2}}$ for all $n \in \mathbb{N}$ and $p \in (0, 1)$. By choosing p and n appropriately, deduce that $R(4, t) = \Omega((t/\log t)^{\frac{3}{2}})$.
7. Find the eigenvalues of the complete r -partite graph $K_r(t)$ and of the cycle C_n .
8. Let G be a connected graph of maximal degree Δ .
(a) Show that if $-\Delta$ is an eigenvalue of G then G is bipartite.
(b) Show that if G is bipartite and μ is an eigenvalue of G then $-\mu$ is also an eigenvalue of G , and that μ and $-\mu$ have the same multiplicity.
9. Let G be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique 4-cycle. Show that $|G| \in \{3, 9, 99, 243, 6273, 494019\}$.
10. Any two members of a certain College have a unique common enemy who is also a member of the College. Show that there is some College member (the ‘Junior Bursar’) who is everyone else’s enemy.