Graph Theory (2013–14) Example Sheet 4 of 4

1. Prove that $R(3,3) = 6$ and $R(3,4) = 9$. By considering the graph with vertex set $[17]$ in which x is joined to y if $x - y$ is a square (modulo 17), show that $R(4,4) = 18$.

2. Prove that $R_k(3,3,\ldots 3) \leq \lfloor ek! \rfloor + 1$. Show that if the numbers $1, 2, \ldots, \lfloor ek! \rfloor$ are partitioned into k classes then the equation $x + y = z$ is soluble in some class. Infer that, for fixed n, the "Fermat" equation $x^n + y^n \equiv z^n \pmod{p}$ has a non-trivial solution (that is, $xyz \not\equiv 0$) for all sufficiently large primes p. (Schur, 1916)

3. Let $A \subset \mathbb{R}^2$ be finite, with no three points collinear. Show that if $|A| \geq R^{(4)}(n,5)$ then A contains n points forming a convex n-gon. Prove the same if $|A| \geq R^{(3)}(n,n)$.

4. Let $r(G)$ be the smallest n such that every red-blue colouring of the edges of K_n contains a monochromatic copy of the graph G. (Note that $r(G)$ exists because $r(G) \leq R(|G|)$.)
Let I_k be a set of k independent edges, so $|I_k| = 2k$. Show that $r(I_k) = 3k - 1$.
Let H_k consist of a triangle xyz and k edges $xx_1, xx_2, \ldots, xx_k$, so $|H_k| = k + 3$. Show that $r(H_1) = 7$. What is $r(H_k)$?

5. Exhibit a 2-colouring of the edges of $K_{(s-1)^2}$ containing no monochromatic K_s.
Colour the edges of the complete graph with vertex set $[s-1]^{(3)}$ so that AB is red if $|A \cap B| = 1$ and AB is blue otherwise. Show that there is no monochromatic K_s.

6. Let the infinite subsets of $\mathbb{N}$ be 2-coloured. Must there exist an infinite set $M \subset \mathbb{N}$ all of whose infinite subsets have the same colour?

7. By painting its vertices red or blue at random, show that a graph G has a bipartition $V(G) = V_1 \cup V_2$ such that $e(G[V_1]) + e(G[V_2]) \leq \frac{1}{2}e(G)$. (c.f. Example Sheet 1)

8. Let $p \in (0,1)$ be fixed. Show that $G \in \mathcal{G}(n,p)$ has diameter 2 whp.

9. In a *tournament* on n players, each pair play a game, with one or other player winning (there are no draws). Prove that, for any k, there is a tournament in which, for any k players, there is a player who beats all of them. Exhibit such a tournament for $k = 2$.

10. Show that for every $n \geq 1$ there is an $n \times n$ bipartite graph of size at least $\frac{1}{2}n^{2-\sigma}$ which contains no $K_{s,t}$, where $\sigma = (s+t-2)/(st-1)$.

11. Show that $R(s,t) > n - \binom{n}{s}p^{\binom{s}{2}} - \binom{n}{t}(1-p)^{\binom{t}{2}}$ for every n and p. By taking $p = n^{-2/3}$, deduce that $R(4,t) > (t/3\log t)^{3/2}$ for large t.

12. Let X be the number of K_4's in $G \in \mathcal{G}(n,p)$. Show that $\mathrm{E}X = \binom{n}{4}p^6$ and that $\mathrm{Var}X/\mathrm{E}X = (1-p^6) + 4(n-4)(p^3-p^6) + 6\binom{n-4}{2}(p^5-p^6)$. Hence show that if $p/n^{-2/3} \to 0$ then $X = 0$ whp, whereas if $p/n^{-2/3} \to \infty$ then $X \neq 0$ whp.

13. Let X be the number of vertices of degree 1 in $G \in \mathcal{G}(n,p)$. Show that $\mathrm{E}X = n(n-1)p(1-p)^{n-2}$ and that, if $\omega(n) \to \infty$ and $p = (\log n + \log\log n + \omega(n))/n$, then $X = 0$ whp. Compute $\mathrm{Var}X$ and show that if $p = \log n/n$ then $X \neq 0$ almost surely.

$^{+}$**14.** Let A be an uncountable set, and let $A^{(2)}$ be 2-coloured. Must there exist an uncountable monochromatic set in A?