

1. Prove that  $R(3, 3) = 6$  and  $R(3, 4) = 9$ . By considering the graph with vertex set  $[17]$  in which  $x$  is joined to  $y$  if  $x - y$  is a square (modulo 17), show that  $R(4, 4) = 18$ .
2. Prove that  $R_k(3, 3, \dots, 3) \leq \lfloor ek! \rfloor + 1$ . Show that if the numbers  $1, 2, \dots, \lfloor ek! \rfloor$  are partitioned into  $k$  classes then the equation  $x + y = z$  is soluble in some class. Infer that, for fixed  $n$ , the “Fermat” equation  $x^n + y^n \equiv z^n \pmod{p}$  has a non-trivial solution (that is,  $xyz \not\equiv 0$ ) for all sufficiently large primes  $p$ . (Schur, 1916)
3. Let  $A \subset \mathbb{R}^2$  be finite, with no three points collinear. Show that if  $|A| \geq R^{(4)}(n, 5)$  then  $A$  contains  $n$  points forming a convex  $n$ -gon. Prove the same if  $|A| \geq R^{(3)}(n, n)$ .
4. Let  $r(G)$  be the smallest  $n$  such that every red-blue colouring of the edges of  $K_n$  contains a monochromatic copy of the graph  $G$ . (Note that  $r(G)$  exists because  $r(G) \leq R(|G|)$ .)  
Let  $I_k$  be a set of  $k$  independent edges, so  $|I_k| = 2k$ . Show that  $r(I_k) = 3k - 1$ .  
Let  $H_k$  consist of a triangle  $xyz$  and  $k$  edges  $xx_1, xx_2, \dots, xx_k$ , so  $|H_k| = k + 3$ . Show that  $r(H_1) = 7$ . What is  $r(H_k)$ ?
5. Exhibit a 2-colouring of the edges of  $K_{(s-1)^2}$  containing no monochromatic  $K_s$ .  
Colour the edges of the complete graph with vertex set  $[s - 1]^{(3)}$  so that  $AB$  is red if  $|A \cap B| = 1$  and  $AB$  is blue otherwise. Show that there is no monochromatic  $K_s$ .
6. Let the infinite subsets of  $\mathbb{N}$  be 2-coloured. Must there exist an infinite set  $M \subset \mathbb{N}$  all of whose infinite subsets have the same colour?
7. By painting its vertices red or blue at random, show that a graph  $G$  has a bipartition  $V(G) = V_1 \cup V_2$  such that  $e(G[V_1]) + e(G[V_2]) \leq \frac{1}{2}e(G)$ . (c.f. Example Sheet 1)
8. Let  $p \in (0, 1)$  be fixed. Show that  $G \in \mathcal{G}(n, p)$  has diameter 2 whp.
9. In a *tournament* on  $n$  players, each pair play a game, with one or other player winning (there are no draws). Prove that, for any  $k$ , there is a tournament in which, for any  $k$  players, there is a player who beats all of them. Exhibit such a tournament for  $k = 2$ .
10. Show that for every  $n \geq 1$  there is an  $n \times n$  bipartite graph of size at least  $\frac{1}{2}n^{2-\sigma}$  which contains no  $K_{s,t}$ , where  $\sigma = (s + t - 2)/(st - 1)$ .
11. Show that  $R(s, t) > n - \binom{n}{s}p^{\binom{s}{2}} - \binom{n}{t}(1-p)^{\binom{t}{2}}$  for every  $n$  and  $p$ . By taking  $p = n^{-2/3}$ , deduce that  $R(4, t) > (t/3 \log t)^{3/2}$  for large  $t$ .
12. Let  $X$  be the number of  $K_4$ 's in  $G \in \mathcal{G}(n, p)$ . Show that  $\mathbb{E}X = \binom{n}{4}p^6$  and that  $\text{Var}X/\mathbb{E}X = (1 - p^6) + 4(n - 4)(p^3 - p^6) + 6\binom{n-4}{2}(p^5 - p^6)$ . Hence show that if  $p/n^{-2/3} \rightarrow 0$  then  $X = 0$  whp, whereas if  $p/n^{-2/3} \rightarrow \infty$  then  $X \neq 0$  whp.
13. Let  $X$  be the number of vertices of degree 1 in  $G \in \mathcal{G}(n, p)$ . Show that  $\mathbb{E}X = n(n - 1)p(1 - p)^{n-2}$  and that, if  $\omega(n) \rightarrow \infty$  and  $p = (\log n + \log \log n + \omega(n))/n$ , then  $X = 0$  whp. Compute  $\text{Var}X$  and show that if  $p = \log n/n$  then  $X \neq 0$  almost surely.
- + 14. Let  $A$  be an uncountable set, and let  $A^{(2)}$  be 2-coloured. Must there exist an uncountable monochromatic set in  $A$ ?