

1. Let  $G$  be a graph and  $v \in G$ . Must  $\kappa(G - v) \leq \kappa(G)$ ?
2. Show that, for any graph  $G$ ,  $\kappa(G) \leq \lambda(G)$ , and that if  $G$  is 3-regular then  $\kappa(G) = \lambda(G)$ . Given positive integers  $k \leq \ell$ , construct a graph  $G$  with  $\kappa(G) = k$  and  $\lambda(G) = \ell$ .
3. Let  $G$  be a bipartite graph with vertex classes  $X$  and  $Y$ . Show that if  $G$  has a matching from  $X$  to  $Y$  then there exists  $x \in X$  such that every edge containing  $x$  extends to a matching from  $X$  to  $Y$ .
4. For each  $n \geq 4$ , let  $G_n$  be the graph with vertex set  $[n]$ , and  $ij$  an edge iff  $i - j \equiv \pm 1$  or  $\pm 2 \pmod n$ . For which  $n$  is  $G_n$  planar?
5. An  $n \times n$  *Latin square* (resp.  $r \times n$  *Latin rectangle*) is an  $n \times n$  (resp.  $r \times n$ ) matrix, with entries in  $[n]$ , such that no two entries in the same row or column are the same. Prove that every  $r \times n$  Latin rectangle may be extended to an  $n \times n$  Latin square.
6. Let  $G$  be a  $k$ -connected graph. Suppose that  $v \in G$  and  $U \subset V(G) - \{v\}$  with  $|U| \geq k$ . Show that  $G$  contains  $k$   $vU$ -paths any two of which have only the vertex  $v$  in common.
7. Let  $G$  be a  $k$ -connected graph ( $k \geq 2$ ), and let  $x_1, x_2, \dots, x_k$  be vertices of  $G$ . Show that there is a cycle in  $G$  containing all the  $x_i$ .
8. Show, without assuming the four-colour theorem, that every triangle-free planar graph is four-colourable.
9. Let  $(G_n)$  be a sequence of graphs with  $|G_n| = n$  for each  $n$ . If there is some  $\varepsilon > 0$  such that we have  $e(G_n) \geq (\frac{2}{3} + \varepsilon) \binom{n}{2}$  for every  $n$ , why must every planar graph be a subgraph of some  $G_n$ ? Show that this need not be the case if instead  $e(G_n)/\binom{n}{2} \rightarrow 2/3$ .
10. Find the chromatic polynomial of the  $n$ -cycle.
11. Let  $G$  be a graph with chromatic polynomial  $p(X) = X^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \dots + a_0$ . Show that the coefficients  $a_i$  alternate in sign. Show also that if  $G$  has  $m$  edges and  $t$  triangles then  $a_{n-2} = \binom{m}{2} - t$ .
12. What is  $\chi'(K_{n,n})$ ? What is  $\chi'(K_n)$ ?
13. Show that every  $\Delta$ -regular bipartite graph is  $\Delta$ -edge-colourable. Is it true that for every bipartite graph  $G$  we have  $\chi'(G) = \Delta(G)$ ?
14. Let  $G$  be an infinite bipartite graph, with vertex classes  $X$  and  $Y$  such that  $|\Gamma(A)| \geq |A|$  for every  $A \subset X$ . Give an example to show that  $G$  need not contain a matching from  $X$  to  $Y$ . Show however that if  $G$  is countable and every vertex in  $X$  has finite degree then  $G$  does contain a matching from  $X$  to  $Y$ . Does this remain true if  $G$  is uncountable?
15. Let  $A$  be an  $n \times n$  *doubly stochastic* matrix; that is, its entries are non-negative real numbers and each of its rows and columns sums to one. Show that  $A$  is in the convex hull of the set of  $n \times n$  permutation matrices; that is, show that there are permutation matrices  $P_1, P_2, \dots, P_m$  and non-negative reals  $\lambda_1, \lambda_2, \dots, \lambda_m$  with  $\sum_{i=1}^m \lambda_i = 1$  such that  $A = \sum_{i=1}^m \lambda_i P_i$ . How small can one choose  $m$ ?