

1. Let G be a graph and $v \in G$. Must $\kappa(G - v) \leq \kappa(G)$?
2. Show that, for any graph G , $\kappa(G) \leq \lambda(G)$, and that if G is 3-regular then $\kappa(G) = \lambda(G)$. Given positive integers $k \leq \ell$, construct a graph G with $\kappa(G) = k$ and $\lambda(G) = \ell$.
3. Let G be a bipartite graph with vertex classes X and Y . Show that if G has a matching from X to Y then there exists $x \in X$ such that every edge containing x extends to a matching from X to Y .
4. For each $n \geq 4$, let G_n be the graph with vertex set $[n]$, and ij an edge iff $i - j \equiv \pm 1$ or $\pm 2 \pmod{n}$. For which n is G_n planar?
5. An $n \times n$ *Latin square* (resp. $r \times n$ *Latin rectangle*) is an $n \times n$ (resp. $r \times n$) matrix, with entries in $[n]$, such that no two entries in the same row or column are the same. Prove that every $r \times n$ Latin rectangle may be extended to an $n \times n$ Latin square.
6. Let G be a k -connected graph. Suppose that $v \in G$ and $U \subset V(G) - \{v\}$ with $|U| \geq k$. Show that G contains k vU -paths any two of which have only the vertex v in common.
7. Let G be a k -connected graph ($k \geq 2$), and let x_1, x_2, \dots, x_k be vertices of G . Show that there is a cycle in G containing all the x_i .
8. Show, without assuming the four-colour theorem, that every triangle-free planar graph is four-colourable.
9. Let (G_n) be a sequence of graphs with $|G_n| = n$ for each n . If there is some $\varepsilon > 0$ such that we have $e(G_n) \geq (\frac{2}{3} + \varepsilon) \binom{n}{2}$ for every n , why must every planar graph be a subgraph of some G_n ? Show that this need not be the case if instead $e(G_n)/\binom{n}{2} \rightarrow 2/3$.
10. Find the chromatic polynomial of the n -cycle.
11. Let G be a graph with chromatic polynomial $p(X) = X^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \dots + a_0$. Show that the coefficients a_i alternate in sign. Show also that if G has m edges and t triangles then $a_{n-2} = \binom{m}{2} - t$.
12. What is $\chi'(K_{n,n})$? What is $\chi'(K_n)$?
13. Show that every Δ -regular bipartite graph is Δ -edge-colourable. Is it true that for every bipartite graph G we have $\chi'(G) = \Delta(G)$?
14. Let G be an infinite bipartite graph, with vertex classes X and Y such that $|\Gamma(A)| \geq |A|$ for every $A \subset X$. Give an example to show that G need not contain a matching from X to Y . Show however that if G is countable and every vertex in X has finite degree then G does contain a matching from X to Y . ⁺Does this remain true if G is uncountable?
15. Let A be an $n \times n$ *doubly stochastic* matrix; that is, its entries are non-negative real numbers and each of its rows and columns sums to one. Show that A is in the convex hull of the set of $n \times n$ permutation matrices; that is, show that there are permutation matrices P_1, P_2, \dots, P_m and non-negative reals $\lambda_1, \lambda_2, \dots, \lambda_m$ with $\sum_{i=1}^m \lambda_i = 1$ such that $A = \sum_{i=1}^m \lambda_i P_i$. How small can one choose m ?