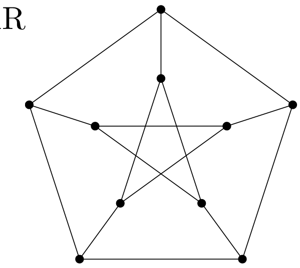
Mich. 2011 **GRAPH THEORY—EXAMPLES 2** PAR



1. For which m and n is the complete bipartite graph $K_{m,n}$ Hamiltonian?

2. What is $\lim_{n\to\infty} \operatorname{ex}(n; P)/\binom{n}{2}$, where P is the Petersen graph (shown)?

3. For each $r \geqslant 3$, construct a graph of chromatic number r that contains no K_r.

4. Without assuming Turán's theorem, show that the Turán graph $T_r(n)$ is the unique r-partite graph of order n with the largest possible number of edges.

5. Let G be a graph of order n. Suppose that $d(x)+d(y) \geqslant n$ for all distinct, non-adjacent x, $y \in G$. Show that G is Hamiltonian. Deduce that any graph of order $n \geqslant 3$ with $\binom{n}{2} - n + 3$ edges must be Hamiltonian.

6. Show that $\operatorname{ex}(n; K_{2,t}) \leqslant \frac{n}{4}\left(1 + \sqrt{1 + 4(n-1)(t-1)}\right)$.

7. What is $\operatorname{ex}(n; K_{1,t})$?

8. Let G be a graph of order $n \geqslant 5$ with $e(G) \geqslant \lfloor \frac{n^2}{4} \rfloor + 2$. Show that G must contain two triangles with precisely one vertex in common.

9. Let $x_1, x_2, \ldots, x_{3n}$ be points in the plane such that no two of them are more than distance 1 apart. Prove that at most $3n^2$ of the distances $\|x_i - x_j\|$ $(i < j)$ are greater than $1/\sqrt{2}$.

10. For each positive integer n, let g_n be the largest integer k such that it is possible to colour k edges of the complete graph K_n blue or yellow without creating a monochromatic (blue or yellow) triangle. Show that $g_n/\binom{n}{2}$ converges, and find $\lim_{n\to\infty} g_n/\binom{n}{2}$.

11. How many edges can a graph G of order n containing precisely one triangle have?

12. Let $x_1, x_2, \ldots \in \mathbb{R}^d$ be distinct and, for each n, let $f(n)$ be the number of pairs of points from $x_1, \ldots, x_n$ that are at distance 1: $f(n) = |\{(i,j) : 1 \leqslant i < j \leqslant n, \|x_i - x_j\| = 1\}|$. Show that if $d = 3$ then $f(n)/\binom{n}{2} \to 0$ as $n \to \infty$. Show more generally that if $d \geqslant 3$ then $f(n) \leqslant (1 - \frac{1}{D} + o(1))\binom{n}{2}$, where $D = \lfloor d/2 \rfloor$.

$^+$13. Construct a triangle-free graph of chromatic number 1526.

$^+$14. Show that an r-regular graph of order $2r+1$ must be Hamiltonian.