Mich. 2010 **GRAPH THEORY—EXAMPLES 3** PAR

1. Show that for any graph G we have $e(G) \geq \binom{\chi(G)}{2}$.

2. For each $n \geq 1$, let G_n be the graph with vertex set $[n]$, and ij an edge iff $i - j \equiv \pm 1$ or $\pm 2 \pmod n$. For which n is G_n planar?

3. Show, without assuming the four-colour theorem, that every triangle-free planar graph is four-colourable.

4. Let (G_n) be a sequence of graphs with $|G_n| = n$ for each n. If there is some $\varepsilon > 0$ such that we have $e(G_n) \geq (\frac{2}{3} + \varepsilon)\binom{n}{2}$ for every n, why must every planar graph be a subgraph of some G_n? Show that this need not be the case if instead $e(G_n)/\binom{n}{2} \to 2/3$.

5. Assuming the existence of triangle-free graphs of arbitrarily large chromatic number, show that there exists a sequence (G_n) of graphs such that $\chi(G_n) \to \infty$ but such that the sequences $(\omega(G_n))$ and $(|G_n|/\alpha(G_n))$ are *both* bounded.

6. What is $\chi'(K_{n,n})$? What is $\chi'(K_n)$?

7. Find the chromatic polynomial of the n-cycle.

8. Let G be a graph with chromatic polynomial $p(X) = X^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_0$. Show that the coefficients a_i alternate in sign. Show also that if G has m edges and t triangles then $a_{n-2} = \binom{m}{2} - t$.

9. Show that every Δ-regular bipartite graph is Δ-edge-colourable. Is it true that for every bipartite graph G we have $\chi'(G) = \Delta(G)$?

10. Let G be a bipartite graph with vertex classes X and Y. Show that if G has a matching from X to Y then there exists $x \in X$ such that every edge containing x extends to a matching from X to Y.

11. An $n \times n$ *Latin square* (resp. $r \times n$ *Latin rectangle*) is an $n \times n$ (resp. $r \times n$) matrix, with entries in $[n]$, such that no two entries in the same row or column are the same. Prove that every $r \times n$ Latin rectangle may be extended to an $n \times n$ Latin square.

12. Let G be an infinite bipartite graph, with vertex classes X and Y such that $|\Gamma(A)| \geq |A|$ for every $A \subset X$. Give an example to show that G need not contain a matching from X to Y . Show however that if G is countable and every vertex in X has finite degree then G does contain a matching from X to Y. $^+$Does this remain true if G is uncountable?

13. Let G be a finite group, and H a subgroup of G of index k. Show that there must exist some elements $g_1, g_2, \ldots, g_k \in G$ such that $g_1H, g_2H, \ldots, g_kH$ are the left cosets of H and $Hg_1, Hg_2, \ldots, Hg_k$ are the right cosets of H.

14. A *chain* in $\mathcal{P}[n] = \{A : A \subset [n]\}$ is a sequence $A_0 \subset A_1 \subset \cdots \subset A_k$ of subsets of $[n]$. Show that $\mathcal{P}[n]$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ disjoint chains, but cannot be partitioned into $\binom{n}{\lfloor n/2 \rfloor} - 1$ disjoint chains.

15. Let G be a planar graph (so we know that G has a drawing in the plane in which each edge is polygonal). Show that there is a drawing of G in the plane in which each edge is represented by a straight line.