Mich. 2009 **GRAPH THEORY—EXAMPLES 4** PAR

1. Let $G = (V, E)$ be a digraph such that $\Gamma^+(v) \neq \emptyset$ for all $v \in G$. Show that G contains a (directed) cycle, i.e. distinct vertices $v_1, v_2, \ldots, v_k$ with $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, k-1$ and $v_k v_1 \in E$.

2. Why does the Ford-Fulkerson algorithm still terminate if we allow the capacities of edges to be rational, rather than just integral? What if we also allow the capacities to be irrational?

3. Let G be a graph and $v \in G$. Must $\kappa(G - v) \leq \kappa(G)$?

4. For each $n \geq 3$, show that there is a tournament of order n containing $2^{-n}(n-1)!$ (directed) Hamilton cycles.

5. Let $p \in (0, 1)$ be fixed. Show that almost every $G \in \mathcal{G}(n, p)$ has diameter 2.

6. Show that, for any graph G, $\kappa(G) \leq \lambda(G)$. Given positive integers $k \leq \ell$, construct a graph G with $\kappa(G) = k$ and $\lambda(G) = \ell$.

7. Show that if G is 3-regular then $\kappa(G) = \lambda(G)$.

8. For a set $B \subset V(G)$ and a vertex a not in B, an *a-B fan* is a family of $|B|$ paths from a to B, any two meeting only at a. Show that a graph G (with $|G| > k$) is k-connected if and only if there is an a-B fan for every $B \subset V(G)$ with $|B| = k$ and every vertex a not in B.

9. Let G be a k-connected graph ($k \geq 2$), and let $x_1, x_2, \ldots, x_k$ be vertices of G. Show that there is a cycle in G containing all the x_i.

10. Prove that, for any k, there is a tournament in which, for any k players, there is a player who beats all of them. Exhibit such a tournament for $k = 2$.

11. Show that a tournament must contain a (directed) Hamilton path, i.e. a directed path through all the vertices. Must it contain a (directed) Hamilton cycle?

12. Let X denote the number of copies of K_4 in a random graph G chosen from $G(n, p)$. Find the mean and the variance of X. Deduce that $p = n^{-2/3}$ is a threshold for the existence of a K_4, in the sense that if $pn^{2/3} \to 0$ then almost surely G does not contain a K_4, while if $pn^{2/3} \to \infty$ then almost surely G does contain a K_4.

13. Find the eigenvalues of the graph K_n, and of the graph $K_{m,n}$.

14.Let G be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique 4-cycle. Show that $|G| \in \{3, 9, 99, 243, 6273, 494019\}$.

$^+$15. At a certain conference, it transpires that each pair of participants have precisely one co-author in common, and that their common co-author is also attending the conference. Show that there is someone at the conference ('Erdős') who is a co-author of every other participant.