Mich. 2009 **GRAPH THEORY—EXAMPLES 1** PAR

1. Show that every graph (of order at least 2) has two vertices of the same degree.

2. Construct a 3-regular graph on 8 vertices. Is there a 3-regular graph on 9 vertices?

3. A graph G is *self-complementary* if it is isomorphic to its complement. Show that there exists a self-complementary graph of order n if and only if $n \equiv 0$ or $1 \pmod 4$.

4. Show that every graph of average degree d contains a subgraph of minimum degree at least $d/2$.

5. Let G be a regular graph of order n containing $K_{\lceil (n+1)/2 \rceil}$ as a subgraph. Show that $G = K_n$.

6. Let G be a graph. Show that its vertex set V has a partition $V = V_1 \cup V_2$ such that $e(G[V_1]) + e(G[V_2]) \leq \frac{1}{2}e(G)$. Show that one may demand in addition that each V_i span at most a third of the edges; that is, $e(G[V_i]) \leq \frac{1}{3}e(G)$ for $i = 1, 2$.

7. Show that $R(3,4) = 9$ and $R(4) = 18$. *[Hint: consider the graph with vertex-set* $[17]$*, where* ij *is an edge iff* $i - j$ *is a square modulo 17.]*

8. Let $f_1, f_2, \ldots, f_n\colon \mathbb{R} \to \mathbb{R}$ be bounded functions and let $\delta, \varepsilon > 0$. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is such that, whenever we have $x, y \in \mathbb{R}$ with $|f(x) - f(y)| > \delta$, then $|f_i(x) - f_i(y)| > \varepsilon$ for some i. Show that f is bounded.

9. Show that there is an infinite set S of positive integers such that the sum of any two distinct elements of S has an even number of distinct prime factors.

10. Given a graph G, let $R(G)$ be the smallest n such that every red-blue colouring of K_n yields a monochromatic copy of G.

(a) How do we know that $R(G)$ exists?

(b) Let I_k be a set of k independent edges (so $|I_k| = 2k$). Show that $R(I_k) = 3k - 1$.

(c) Let H_k consist of a triangle xyz and k edges $xx_1, xx_2, \ldots, xx_k$ (so $|H_k| = k+3$). Show that $R(H_1) = 7$. What is $R(H_k)$?

(d) Show that $R(C_4) = 6$.

11. Show that $R_k(s) \leq 4^{s^{k-1}}$. By giving a two-pass proof of the multicolour Ramsey theorem, or otherwise, show that in fact $R_k(s) \leq k^{ks}$.

12. Let k be a positive integer. Must there exist a positive integer n such that, if we partition $[n]$ into k parts, $[n] = A_1 \cup \cdots \cup A_k$, then we can find i and *distinct* $a, b, c \in A_i$ with $a + b = c$?

13. Let n be a positive integer. Show that, for any sufficiently large prime p, there exist $x, y, z, w \in (\mathbb{Z}/p\mathbb{Z})^\times$ with $x^n + y^n + z^n = w^n$. State explicitly what is meant by 'sufficiently large'; that is, give some number N (depending on n) such that the result holds for all primes $p \geq N$.

14. (a) For each integer $s \geq 3$, exhibit a 2-colouring of the edges of the graph $K_{(s-1)^2}$ containing no monochromatic K_s.

(b) Let $\mathcal{A}$ be a collection of subsets of $[s-1]$. Suppose that (i) $|A| = 3$ for each $A \in \mathcal{A}$; and (ii) $|A \cap B| = 1$ for all distinct $A, B \in \mathcal{A}$. Show that $|\mathcal{A}| \leq s - 1$. Show also that the same holds if we replace the condition '$|A \cap B| = 1$' in (ii) with '$|A \cap B| \neq 1$'. Hence exhibit a 2-colouring of the edges of $K_{\binom{s-1}{3}}$ containing no monochromatic K_s.