Mich. 2008

1. What is the chromatic number of the Petersen graph? What is its edge-chromatic number?

2. What is  $\chi'(K_{n,n})$ ? What is  $\chi'(K_n)$ ?

3. Let G be a graph with chromatic number k. Show that  $e(G) \ge \binom{k}{2}$ .

4. Show that, for any graph G, there is an ordering of the vertices of G for which the greedy algorithm uses only  $\chi(G)$  colours.

5. For each  $k \ge 3$ , find a bipartite graph G, with an ordering  $v_1, v_2, \ldots, v_n$  of its vertices, for which the greedy algorithm uses k colours. Give an example with n = 2k - 2. Is there an example with n = 2k - 3?

6. Let G be a bipartite graph of maximum degree  $\Delta$ . Must we have  $\chi'(G) = \Delta$ ?

7. Find the chromatic polynomial of the n-cycle.

8. Let G be a graph on n vertices, with  $p_G(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \ldots + a_1t + a_0$ . Show that the  $a_i$  alternate in sign (in other words,  $a_i \leq 0$  if n-i is odd and  $a_i \geq 0$  if n-i is even). Show also that if G has m edges and c triangles then  $a_{n-2} = \binom{m}{2} - c$ .

9. An acyclic orientation of a graph G is an assignment of a direction to each edge of G in such a way that there is no directed cycle. Show that the number of acyclic orientations of G is precisely  $|p_G(-1)|$ .

10. Let G be a plane graph in which every face is a triangle. Show that the faces of G may be 3-coloured, unless  $G = K_4$ .

11. Can  $K_{4,4}$  be drawn on the torus? What about  $K_{5,5}$ ?

12. A minor of a graph G is any graph that may be obtained from a subgraph of G by successively contracting edges – equivalently, a graph H on vertex-set  $\{v_1, \ldots, v_r\}$  is a minor of G if we can find disjoint connected subgraphs  $S_1, \ldots, S_r$  of G such that whenever  $v_i v_j \in E(H)$  there is an edge from  $S_i$  to  $S_j$ . Show that for any k there is an n such that every graph G with  $\chi(G) \ge n$  has a  $K_k$  minor. Writing c(k) for the least such n, show that  $c(k+1) \le 2c(k)$ . [Hint: choose  $x \in G$ , and look at the sets  $\{y \in G : d(x, y) = t\}$ .] Show that c(k) = k for  $1 \le k \le 4$ , and explain why c(5) = 5 would imply the 4-Colour Theorem.

13. Let G be a countable graph in which every finite subgraph can be k-coloured. Show that G can be k-coloured.

+14. Construct a triangle-free graph of chromatic number 1526.