

1. Show that  $R(3, 4) \leq 9$ . By considering the graph on  $\mathbb{Z}_8$  (the integers modulo 8) in which  $x$  is joined to  $y$  if  $x - y = \pm 1$  or  $\pm 2$ , show that  $R(3, 4) = 9$ .
2. By considering the graph on  $\mathbb{Z}_{17}$  in which  $x$  is joined to  $y$  if  $x - y$  is a square modulo 17, show that  $R(4, 4) = 18$ .
3. Show that  $R_3(3, 3, 3) \leq 17$ .
4. Let  $A$  be a set of  $R^{(4)}(n, 5)$  points in the plane, with no three points of  $A$  collinear. Prove that  $A$  contains  $n$  points forming a convex  $n$ -gon.
5. Let  $A$  be an infinite set of points in the plane, with no three points of  $A$  collinear. Prove that  $A$  contains an infinite set  $B$  such that no point of  $B$  is a convex combination of other points of  $B$ .
6. Show that every graph  $G$  has a partition of its vertex-set as  $X \cup Y$  such that the number of edges from  $X$  to  $Y$  is at least  $\frac{1}{2}e(G)$ . Give three proofs: by induction, by choosing an optimal partition, and by choosing a random partition.
7. In a *tournament* on  $n$  players, each pair play a game, with one or other player winning (there are no draws). Prove that, for any  $k$ , there is a tournament in which, for any  $k$  players, there is a player who beats all of them. [Hint: consider a random tournament.] Exhibit such a tournament for  $k = 2$ .
8. Let  $X$  denote the number of copies of  $K_4$  in a random graph  $G$  chosen from  $G(n, p)$ . Find the mean and the variance of  $X$ . Deduce that  $p = n^{-2/3}$  is a threshold for the existence of a  $K_4$ , in the sense that if  $pn^{2/3} \rightarrow 0$  then almost surely  $G$  does not contain a  $K_4$ , while if  $pn^{2/3} \rightarrow \infty$  then almost surely  $G$  does contain a  $K_4$ .
9. Find the eigenvalues of  $K_n$ . Find the eigenvalues of  $K_{n,m}$ .
10. Prove that the matrix  $J$  (all of whose entries are 1) is a polynomial in the adjacency matrix of a graph  $G$  if and only if  $G$  is regular and connected.
11. Let  $G$  be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique 4-cycle. Show that  $G$  is  $k$ -regular, for some  $k$ , and that the number of vertices of  $G$  is  $1 + k^2/2$ . Show also that  $k$  must belong to the set  $\{2, 4, 14, 22, 112, 994\}$ .
12. Let the infinite subsets of  $\mathbb{N}$  be 2-coloured. Must there exist an infinite set  $M \subset \mathbb{N}$  all of whose infinite subsets have the same colour?
- +13. Let  $A$  be an uncountable set, and let  $A^{(2)}$  be 2-coloured. Must there exist an uncountable monochromatic set in  $A$ ?