MATHEMATICAL TRIPOS PART II (2006–07)

Graph Theory - Example Sheet 4 of 4 **O.M. Riordan**

Basic Examples: straightforward material on some of the main definitions and theorems.

B1) Prove that $R(3,3) = 6$ and $R(3,4) = 9$. By considering the graph with vertex set [17] in which x is joined to y if $x - y$ is a square (modulo 17), show that $R(4,4) = 18$.

B2) By painting its vertices red or blue at random, show that a graph G has a bipartition $V(G) = V_1 \cup V_2$ such that $e(G[V_1]) + e(G[V_2]) \leq \frac{1}{2}e(G)$. (c.f. example sheet 1).

B3) Let X be the number of K_4's in $G(n,p)$. Show that $\mathbb{E}X = \binom{n}{4}p^6$ and calculate $\mathbb{E}(X^2)$ or Var$X$. Hence show that if $p/n^{-2/3} \to 0$ then $X = 0$ whp, whereas if $p/n^{-2/3} \to \infty$ then $X \neq 0$ whp.

B4) Show that the adjacency matrix $A(K_n)$ of K_n has eigenvalues $n-1$ (once) and -1 ($n-1$ times). Show that $A(K_{s,t})$ has eigenvalues $\pm\sqrt{st}$ (once each) and 0 ($s+t-2$ times), and that the Laplacian of $K_{s,t}$ has eigenvalues 0 and $s+t$ (once each), s ($t-1$ times) and t ($s-1$ times).

Exercises: you needn't do all the basic examples before attempting these.

1) Show that $R_3(3,3,3) \leq 17$. Give an example to show that $R_3(3,3,3) = 17$. [There is a symmetric example, with vertex set [16] and the colour of ij depending only on $i - j$ (modulo 16).]

2) Let A be a set of $R^{(4)}(n,5)$ points in the plane, with no three points of A collinear. Prove that A contains n points forming a convex n-gon.

   Give a different argument to prove the same with $R^{(3)}(n,n)$ in place of $R^{(4)}(n,5)$.

3) Prove that there is a tournament (oriented complete graph — see example sheet 1) of order n containing at least $2^{-n}(n-1)!$ directed Hamiltonian cycles.

4) Given a graph G drawn on the plane (maybe with some edges crossing) let $\xi(G)$ be the number of edge crossings. Let $|G| = n$ and $e(G) = m$. Show that $\xi(G) \geq m - 3n + 6$.

   Improve this when $m \geq 4n$ as follows. Choose a random subset $S \subset V(G)$ by choosing vertices independently with probability $p = 4n/m$. Let $X_S = \xi(G[S]) - e(G[S]) + 3|S|$, so $X_S \geq 0$. Show that $\mathbb{E}X_S = p^4\xi(G) - p^2m + 3pn$. Thus $\xi(G) \geq m^3/64n^2$.

5) Use Stirling's formula to show that $\binom{n}{s}2^{1-\binom{s}{2}} < 1$ if $n = \frac{1-\varepsilon}{e\sqrt{2}}s2^{s/2}$ and s is large. Conclude that $R(s) > (\frac{1}{e\sqrt{2}} + o(1))s2^{s/2}$.

   By removing a vertex from each monochromatic K_s in a random colouring, show that $R(s) > n - \binom{n}{s}2^{1-\binom{s}{2}}$ for every n. Conclude that $R(s) > (1/e + o(1))s2^{s/2}$.

6) Show that for every $n \geq 1$ there is an $n \times n$ bipartite graph of size at least $\frac{1}{2}n^{2-\sigma}$ which contains no $K_{s,t}$, where $\sigma = (s+t-2)/(st-1)$.

7) Let X be the number of vertices of degree 1 in $G(n,p)$. Show that $\mathbb{E}X = n(n-1)p(1-p)^{n-2}$. Deduce that, if $\omega(n) \to \infty$ and $p = (\log n + \log\log n + \omega(n))/n$, then $X = 0$ whp. By considering $\mathbb{E}(X^2)$, show that if $p = \log n/n$ then $X \neq 0$ whp.

8) Let G be a graph in which every edge is in a unique K_3 and every non-edge is the diagonal of a unique C_4. Show that $|G| = 1 + 2t^2$ and so G is strongly regular with parameters $(2t, 1, 2)$, for some $t \in \{1, 2, 7, 11, 56, 497\}$.

9) The Laplacian of a d-regular graph of order n is $L = dI - A$, and $\mu_1 = 0, \mu_2, \ldots, \mu_n$ are its eigenvalues. By considering $\mathrm{tr}A^2$ show that $\max_{i\geq 2} |d - \mu_i| \geq \sqrt{d(n-d)/(n-1)}$.

+10) You are at a party where you know at least as many people as anyone else does. You discover that every two people there have exactly one mutual acquaintance at the party. Prove that you know everybody else.

Further Problems: the selection above covers the course but you might enjoy the ones below too.

F1) Given a finite graph G, let $R(G)$ be the smallest n such that every red-blue colouring of K_n yields a monochromatic copy of G.
(i) Let I_k be a set of k independent edges, so $|I_k| = 2k$. Show that $R(I_k) = 3k - 1$.
(ii) Show that $R(C_4) = 6$.

11) Show that $R_k(3, 3, \ldots 3) \leq \lfloor ek! \rfloor + 1$.
Deduce the following theorem of Schur: if we partition the numbers $1, 2, \ldots, \lfloor ek! \rfloor$ into k classes then the equation $x + y = z$ is soluble in at least one of the classes.
Conclude that, for any fixed n, the "Fermat" equation $x^n + y^n \equiv z^n \pmod{p}$ has a non-trivial solution (that is, $xyz \not\equiv 0$) for all sufficiently large primes p.

F2) Let the infinite subsets of $\mathbb{N}$ be 2-coloured. Must there exist an infinite set $M \subset \mathbb{N}$ all of whose infinite subsets have the same colour?

+F3) Let A be an uncountable set, and let $A^{(2)}$ be 2-coloured. Must there exist an uncountable monochromatic set in A?

F4) Prove that every sequence of $mn + 1$ numbers contains an increasing subsequence of length $m + 1$ or a decreasing sequence of length $n + 1$.

F5) Show that $R(s, t) > n - \binom{n}{s} p^{\binom{s}{2}} - \binom{n}{t} (1-p)^{\binom{t}{2}}$ for every n and p. By taking $p = n^{-2/3}$, deduce that $R(4, t) > (t/3 \log t)^{3/2}$ for large t.

F6) Let $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$ be the characteristic polynomial of the adjacency matrix of G. Show that $a_1 = 0$, $a_2 = -e(G)$ and $a_3 = -2 \times$ the number of K_3's in G.

F7) Let B be an incidence matrix for G. Show that G has $|G| - \mathrm{rank}(B)$ components.

F8) Let G be a graph of order n and size m, and B the incidence matrix of some orientation. Let $\widetilde{B}$ be the $(n-1) \times m$ matrix obtained by deleting some row of B. For each set S of $n - 1$ edges of G let P_S be the corresponding $(n-1) \times (n-1)$ submatrix of $\widetilde{B}$.
(i) Show that $\det P_S = \pm 1$ if S forms a spanning tree and $\det P_S = 0$ otherwise.
(ii) The Cauchy-Binet formula states that $\det \widetilde{B}\widetilde{B}^t = \sum_S \det P_S P_S^t$, where S runs over all subsets of $n - 1$ edges. Deduce that G has $n^{-2}\det(L + J)$ spanning trees.
(iii) By taking $G = K_n$, show there are n^{n-2} labelled trees of order n.