MATHEMATICAL TRIPOS PART II (2006–07)

Graph Theory - Example Sheet 2 of 4 **O.M. Riordan**

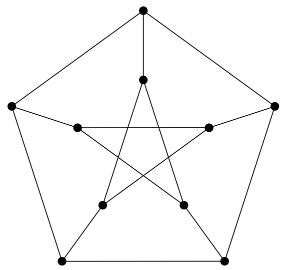
Basic Examples: straightforward material on some of the main definitions and theorems.

B1) Construct a graph of order n with no Hamilton cycle and with size $\binom{n}{2} - (n-2)$. Show that no greater size can be achieved.

B2) Why does every graph G of order n and size $\lfloor n^2/4 \rfloor + 1$ contain a K_3? Show that if $n \geq 6$ it contains a C_5 too. [Hint: How large can $\delta(G)$ be?]

B3) Let G be a graph of order n and size m with $G \not\supset K_{t,t}$. Show that there is an n by n bipartite graph with $2m$ edges not containing $K_{t,t}$, and deduce that $\mathrm{ex}(n, K_{t,t}) \leq \frac{1}{2}((t-1)^{1/t}(n-t+1)n^{1-1/t} + (t-1)n)$.

B4) Show that $\lim_{n\to\infty} \mathrm{ex}(n; P)/\binom{n}{2} = \frac{1}{2}$, where P is the Petersen graph drawn (correctly) below.



Exercises: you needn't do all the basic examples before attempting these.

1) Let G be a graph of order n.
   (i) Let x and y be non-adjacent vertices with $d(x) + d(y) \geq n$. Show that G is Hamiltonian if and only if $G + xy$ is.
   (ii) Form the *closure* $C(G)$ of G by repeatedly joining pairs of vertices whose degree sum is at least n, until no such pairs remain. Show that $C(G)$ is well-defined (that is, the order in which pairs are joined is immaterial).
   (iii) Deduce that, if $d(x) + d(y) \geq n$ whenever $xy \notin E(G)$, then G is Hamiltonian.

2) Show directly, without appealing to Turán's theorem, that if G is a graph of order n with $\delta(G) > \delta(T_r(n)) = n - \lceil n/r \rceil$ then G contains K_{r+1}.

3) Let $v_1, \ldots, v_{3n}$ be vectors in $\mathbb{R}^2$ with $||v_i - v_j|| \leq 1$ for all i, j. Prove that at most $3n^2$ of the distances $||v_i - v_j||$ can, in fact, exceed $1/\sqrt{2}$.
[Hint: Can all the distances amongst four of the points exceed $1/\sqrt{2}$?]

4) Let G have $n \geq r+1$ vertices and $t_{r-1}(n) + 1$ edges.
   (i) Prove that for every p in the range $r \leq p \leq n$, G has a subgraph of order p and size at least $t_{r-1}(p) + 1$.
   (ii) Deduce that if $r \geq 3$ then G contains all but one edge of a K_{r+1}.

5) Prove that for $n \geq 5$ every graph of order n with $\lfloor n^2/4 \rfloor + 2$ edges contains two triangles with exactly one vertex in common.

6) For each p in the range $2 \le p \le n/2$, construct a connected regular graph of order n containing K_p but not K_{p+1}. On the other hand, show that if $p > n/2$ and G is a regular graph of order n containing K_p, then $G = K_n$.

7) Let G be a graph of order n and let $G_1, \dots, G_n$ be the subgraphs of order $n-1$ obtainable by removing a single vertex. Show that $\sum_{i=1}^n e(G_i) = (n-2)e(G)$.
Let F be a graph and let $c_n = \mathrm{ex}(n;F)/\binom{n}{2}$. Deduce that $c_n \le c_{n-1}$ and hence that $\lim_{n\to\infty} c_n$ exists.

8) The *upper density* ud(G) of an infinite graph G is the supremum of the densities of its large finite subgraphs; that is,

$$\mathrm{ud}(G) = \lim_{n\to\infty} \sup\left\{ e(H)/\tbinom{|H|}{2} : H \subset G,\, n \le |H| < \infty \right\}.$$

Show that, for every G, $\mathrm{ud}(G) \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, 1-\frac{1}{r}, \dots, 1\}$.

Further Problems: the selection above covers the course but you might enjoy the ones below too.

F1) Let $A = (a_{ij})_1^n$ be an $n \times n$ doubly stochastic matrix, that is, its entries are non-negative and the rows and columns each sum to one. Show that A is in the convex hull of the set of $n \times n$ permutation matrices, i.e. there are permutation matrices $P_1, P_2, \dots, P_m$ such that $A = \sum_1^m \lambda_i P_i$ and $\sum_1^m \lambda_i = 1$. How small can one choose m?

F2) The *independence number* $\beta(G)$ of a graph is the size of a largest independent vertex subset (spanning no edges). Show that if $\beta(G) \le \kappa(G)$ then G is Hamiltonian.
[Hint: If not, let C be a largest cycle in G and let $x \in V(G) - V(C)$. Find $\kappa(G)$ paths from x to C, and consider the vertices on C preceding the endvertices of these paths.]

$^+$F3) Show that an $r-$regular graph of order $2r+1$ is Hamiltonian.

F4) Let G be a graph of order n and average degree $d = 2e(G)/n$. Let S be the sum $\sum_{uv\in E(G)} d(u) + d(v)$. Show that $nd^2 \le S = \sum_{u\in G} d(u)^2$. Hence deduce that, if $d(u) + d(v) \le 2D$ whenever $uv \in E(G)$, then $d \le D$.
In particular, show that if G satisfies the Ore degree condition, namely $d(u)+d(v) \ge n$ whenever $uv \notin E(G)$, then $e(G) \ge n^2/4$.

F5) Let G be a graph of order n satisfying the *anti*-Ore condition; that is, $d(u) + d(v) \ge n$ whenever $uv \in E(G)$. What is the minimum value of $e(G)$ if $\delta(G) \ge 1$? Can you find the minimum if $\delta(G) = d \le n/2$?

F6) Let q be a prime power and $\mathbb{F}_q$ a field with q elements. The *projective plane* over $\mathbb{F}_q$ has as points all lines in $\mathbb{F}_q^3$ containing the origin, and as lines all planes containing the origin. (A point is on a line if the corresponding line lies in the corresponding plane.) Verify that the projective plane gives rise to an n by n bipartite graph with $\frac{n}{2}(1+\sqrt{4n-3})$ edges containing no $K_{2,2}$, where $n = q^2+q+1$. Show that it also gives rise to a graph with n vertices and $\frac{n}{4}(1+\sqrt{4n-3})$ edges containing no C_4.