MATHEMATICAL TRIPOS PART II (2006–07)

Graph Theory – Example Sheet 1 of 4

O.M. Riordan

Basic Examples: straightforward material on some of the main definitions and theorems.

B0) Show that every graph (of order at least two) has two vertices of the same degree.

B1) Show that every connected graph G (of order at least two) has two vertices u, v such that both $G - u$ and $G - v$ are connected.

B2) Show that every graph can be drawn in $\mathbb{R}^3$ without crossing edges.

B3) Show that every maximal planar graph of order $n \geq 3$ has $3n - 6$ edges.

B4) A graph is *k-regular* if every vertex has degree k. Show that a k-regular bipartite graph has a 1-factor (i.e., a 1-regular spanning subgraph).

B5) Prove that a graph G is k-connected iff $|G| \geq k + 1$ and for any $U \subset V(G)$ with $|U| \geq k$ and for any vertex $x \notin U$, there are k paths from x to U, any pair of paths having only the vertex x in common.

Exercises: you needn't do all the basic examples before attempting these.

1) Let $(d_i)_1^n$ be a sequence of integers, $n \geq 2$. Show that there is a tree with degree sequence $(d_i)_1^n$ if and only if $d_i \geq 1$ for all i and $\sum_{i=1}^n d_i = 2n - 2$.

2) A graph isomorphic to its complement is *self-complementary*. Show that there is a self-complementary graph of order n if and only if $n \equiv 0$ or $1 \pmod 4$.

3) Let G be a graph. Show that its vertex set V has a partition $V = V_1 \cup V_2$ such that

$$e(G[V_1]) + e(G[V_2]) \leq \frac{1}{2} e(G).$$

Show that one may demand in addition that each V_i span at most a third of the edges; that is, $e(G[V_i]) \leq \frac{1}{3} e(G), \quad i = 1, 2.$

4) Prove that every planar graph has a drawing in the plane in which every edge is a straight line segment. [Hint: Apply induction on the order of maximal planar graphs by omitting a suitable vertex.]

5) Let G be a connected, bridgeless plane graph drawn with straight edges. Reprove Euler's formula by evaluating the sum of all angles in all faces of G in two different ways. What can you say if G is drawn on the torus instead of the plane?

6) Let G be an infinite bipartite graph with bipartition $X \cup Y$. Show that Hall's condition does not guarantee a matching from X into Y, but that it does do so if G is countable and every vertex in X has finite degree. What if G is uncountable?

7) Show that $\kappa(G) \leq \lambda(G) \leq \delta(G)$.
Conversely, show that if $1 \leq k \leq \ell \leq d$ are integers, then there is a graph with $\kappa(G) = k$, $\lambda(G) = \ell$ and $\delta(G) = d$.
Construct a graph H with a vertex v such that $\kappa(H) = k$ and $\kappa(H - v) = \ell$.

In the special case that G is *cubic* (all degrees are 3) prove that $\kappa(G) = \lambda(G)$.

8) Prove that if G is k-connected ($k \geq 2$) and $\{x_1, x_2, \ldots, x_k\} \subset V(G)$ then there is a cycle in G of length at least $k+1$ that contains all x_i, $1 \leq i \leq k$.

Further Problems: the selection above covers the course but you might enjoy the ones below too.

F1) Show that there are n^{n-3} trees with n unlabelled vertices and $n-1$ labelled edges.

F2) A *tournament* is a complete *oriented* graph, that is, a complete graph in which each edge uv is given a direction, either from u to v or from v to u. Prove that every tournament contains a directed path containing every vertex.

F3) Let G be a graph of order n, with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n = \Delta$, such that $d_k \geq k$ for $k \leq n - \Delta - 1$. Prove that G is connected.

F4) Show that every forest of order n contains either at least $n/9$ leaves or at least $n/9$ vertex disjoint paths of length 4.

F5) Show that a graph of order n and size m contains at least $m-n+1$ cycles, and that this bound may be attained iff $n - 1 \leq m \leq \lfloor 3(n-1)/2 \rfloor$.

F6) Show that if G is a regular (all degrees equal) bipartite graph then $\kappa(G) \neq 1$.

F7) Let uv be an edge of the graph G. The graph G/uv is obtained by *contracting* the edge uv; that is, the edge uv is removed, u and v are identified and any resulting duplicate edges are deleted. H is a *minor* of G, written $H \prec G$, if it is a subgraph of a graph obtained from G by a sequence of edge-contractions.
   (i) Observe that $G \succ H$ if and only if $V(G)$ contains disjoint subsets W_v, $v \in V(H)$, such that $G[W_v]$ is connected and, whenever $uv \in E(H)$, there is an edge of G between W_u and W_v.
   (ii) Deduce that, if $\Delta(H) \leq 3$, then $G \succ H$ if and only if G contains a subdivision of H.
   (iii) Prove that G is planar if and only if $G \not\succ K_{3,3}$ and $G \not\succ K_5$.

F8) Refer to the previous exercise for the definition of $G \succ H$ and the fact that $G \succ K_4$ if and only if G contains a subdivision of K_4.
   (i) Show that if $\kappa(G) \geq 3$ then $G \succ K_4$.
   (ii) Show that if $G \not\succ K_4$ then G has at least two vertices of degree at most 2.
   (iii) Deduce that if $e(G) \geq 2|G| - 2$ then G contains a subdivision of K_4.

F9) Show that the *Petersen* graph (shown) is non-planar by
   (a) showing it has too many edges
   (b) finding a subdivision of $K_{3,3}$
   (c) finding a K_5 minor.