MATHEMATICAL TRIPOS PART II (2005–06)

Graph Theory - Example Sheet 3 of 4 **A.G. Thomason**

Basic Examples: straightforward material on some of the main definitions and theorems.

B10) Show that $e(G) \geq \binom{\chi(G)}{2}$ holds for every graph G.

B11) Show that, for any graph G, there is an ordering of the vertices on which the greedy colouring algorithm uses only $\chi(G)$ colours.

B12) Let G be a graph of order n with complement $\overline{G}$. Show directly that $n \leq \chi(G)\chi(\overline{G})$ and deduce that $2\sqrt{n} \leq \chi(G) + \chi(\overline{G})$. Prove by induction that $\chi(G) + \chi(\overline{G}) \leq n + 1$.

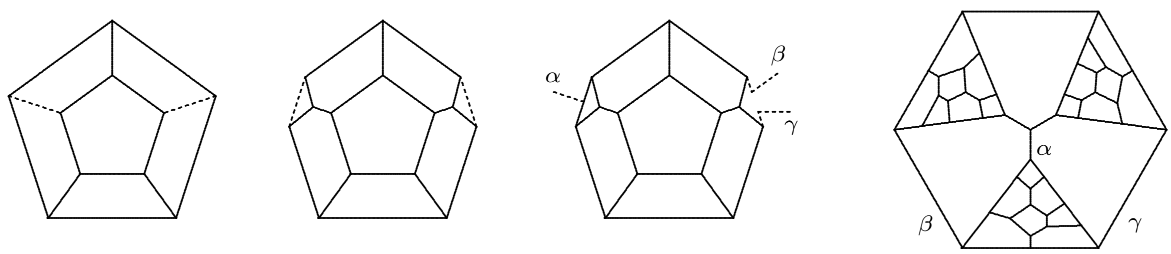
B13) Show directly (i.e., without using the recursive property) that $p_G(x)$ is a polynomial because $p_G(x) = \sum_{i=1}^{n} c_i x(x-1)\dots(x-i+1)$, where c_i is the number of ways to partition G into i non-empty independent (that is, edge-free) sets.

B14) Show that a list colouring of an odd cycle is possible when every vertex has a list of two colours unless the lists are identical.

B15) Draw the maps of the five Platonic solids. What are the dual maps?

B16) Prove that a triangulated plane map is 3-colourable unless it is K_4.

B17) Verify Tutte's counterexample (on right) to Tait's conjecture (maybe the dashes help).



Exercises: you needn't do all the basic examples before attempting these.

17) Let $k \geq 3$ and $n = 2k-2$. Find a bipartite graph with vertices $v_1, v_2, \dots, v_n$ for which the greedy colouring algorithm uses k colours. Is there such a graph with $n = 2k-3$?

18) Let $p_G(x) = \sum_{i=0}^{n} (-1)^i a_i x^{n-i}$ be the chromatic polynomial of G. Prove that $a_i \geq 0$, $a_0 = 1$, $a_1 = e(G)$ and $a_2 = \binom{e(G)}{2} - t(G)$, where $t(G)$ is the number of triangles in G.

19) Find graphs G and H with $|G| = |H|$, $e(G) = e(H)$ and $\chi(G) > \chi(H)$, such that there are more ways to colour G than H when the number of available colours is large.

20) Show that $|p_G(-1)| = (-1)^{|G|} p_G(-1)$ is the number of acyclic orientations of G. (An orientation of G is an assignment of a direction to each edge.)

21) Suppose that G_k is a triangle-free graph with vertex set $\{u_1, \dots, u_n\}$ and satisfying $\chi(G_k) = k$. Construct a graph G_{k+1} from G_k by adding new vertices $\{w, v_1, \dots, v_n\}$ so that v_i is joined to $\Gamma_{G_k}(u_i) \cup \{w\}$. Show that G_{k+1} is triangle-free and that $\chi(G_{k+1}) = k + 1$. Construct explicitly such graphs for $k \leq 4$.

22) Let G be the graph of order 2n+1 obtained by subdividing a single edge of $K_{n,n}$ by a new vertex. Show that $\chi'(G) = \Delta(G) + 1$, but that if e is any edge of G then $\chi'(G - e) = \Delta(G - e)$.

23) Let $G = K_{n,n}$, so $\chi(G) = 2$. If $n \geq \binom{2k-1}{k}$ show that $\chi_\ell(G) \geq k + 1$ by assigning lists of k colours to each vertex so that there is no list colouring.

24) Show that an Eulerian plane map is 2-colourable.

25) Let S be the projective plane. Show that $\max\{\chi(G) : G \text{ embeds on } S\} = 6$.

26) Show that the chromatic number of a triangle-free graph embedded on a surface of Euler characteristic $E \leq 0$ is at most $(5 + \sqrt{25 - 16E})/2$.

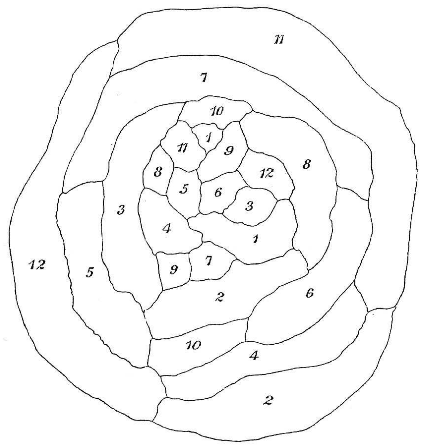
Further Problems: the selection above covers the course but you might enjoy the ones below too.

F16) Show that if G is connected then $(-1)^{|G|-1} p_G(x) > 0$ for all x, $0 < x < 1$.

F17) Let $n = 2^p$. Show that K_{n+1} is not the union of p bipartite graphs but that K_n is. Deduce that among any $2^p + 1$ points in the plane there are three that determine an angle of size at least $\pi(1 - (1/p))$.

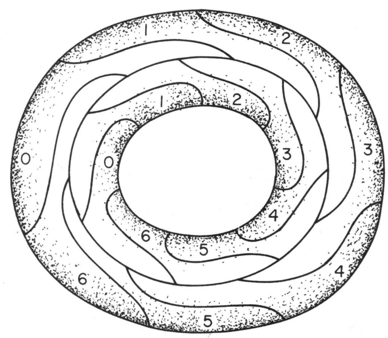
$^+$F18) By examining the proof of Vizing's theorem, show that a graph G with at most two vertices of degree $\Delta(G)$ satisfies $\chi'(G) = \Delta(G)$.

F19) In the colouring of a plane map, an *m-pire* is a set of up to m faces that must receive the same colour (eg France and its overseas departments). Show that if a map has m-pires it can be coloured with $6m$ colours. Find a 2-pire map that needs 12 colours.



left:
Heawood's 2-pire map

right:
Heawood's hoop



F20) Can $K_{4,4}$ be drawn on the torus? Can the 4-cube?
[The n-cube's vertices are all 2^n subsets of $[n]$; join a to b if $|a \triangle b| = 1$.]

F21) Let $u \in G$ and let $U_k = \{v \in G : d(u, v) = k\}$. Show that $\chi(G) \leq \chi(G[U_k]) + \chi(G[U_{k+1}])$ for some k. Deduce that for every natural number p there is a minimal integer $c(p)$ such that every graph G with chromatic number at least $c(p)$ has K_p as a minor (that is, $G \succ K_p$), and indeed $c(1) = 1$, $c(2) = 2$ and $c(p + 1) \leq 2c(p) - 1$ for $p \geq 2$. Show further that $c(3) = 3$ and $c(4) = 4$. (The identity $c(5) = 5$ implies, and can be shown to be equivalent to, the Four Colour Theorem.)

F22) Show: a planar cubic graph is face 3-colourable if and only if each face has even length.