MATHEMATICAL TRIPOS PART II (2005–06)

Graph Theory - Example Sheet 1 of 4 — A.G. Thomason

Basic Examples: straightforward material on some of the main definitions and theorems.

B0) Show that every graph (of order at least two) has two vertices of the same degree.

B1) Show that every connected graph G (of order at least two) has two vertices u, v such that both $G - u$ and $G - v$ are connected.

B2) Show that every graph can be drawn in $\mathbb{R}^3$ without crossing edges.

B3) Show that every maximal planar graph of order $n \geq 3$ has $3n - 6$ edges.

B4) A graph is *k-regular* if every vertex has degree k. Show that a k-regular bipartite graph has a 1-factor (i.e., 1-regular spanning subgraph).

B5) Prove that a graph G is k-connected iff $|G| \geq k + 1$ and for any $U \subset V(G)$ with $|U| \geq k$ and for any vertex $x \notin U$, there are k paths from x to U, any pair of paths having only the vertex x in common.

Exercises: you needn't do all the basic examples before attempting these.

1) Let $(d_i)_1^n$ be a sequence of integers. Show that there is a tree with degree sequence $(d_i)_1^n$ if and only if $d_i \geq 1$ for all i and $\sum_{i=1}^n d_i = 2n - 2$.

2) The *complement* of the graph $G = (V, E)$ is the graph $\overline{G} = (V, V^{(2)} - E)$. A graph isomorphic to its complement is *self-complementary*. Show that there is a self-complementary graph of order n if and only if $n \equiv 0$ or $1 \pmod 4$.

3) Let G be a graph. Show that its vertex set V has a partition $V = V_1 \cup V_2$ such that

$$e(G[V_1]) + e(G[V_2]) \leq \frac{1}{2} e(G).$$

Show also that one may also demand that each V_i span at most a third of the edges; that is, $e(G[V_i]) \leq \frac{1}{3} e(G), \quad i = 1, 2.$

4) Prove that every planar graph has a drawing in the plane in which every edge is a straight line segment. [Hint. Apply induction on the order of maximal planar graphs by omitting a suitable vertex.]

5) Let G be a bipartite graph with bipartition $X \cup Y$ having a matching from X into Y.
   (i) Prove that there is a vertex $x \in X$ such that for every edge xy there is a matching from X to Y that contains xy.
   (ii) Deduce that if $|X| = m$ and $d(x) = d$ for every $x \in X$ then G contains at least $d!$ matchings if $d \leq m$ and at least $d(d-1)\ldots(d-m+1)$ matchings if $d > m$.

6) Show that Hall's condition will not guarantee a matching in an infinite bipartite graph, but that it will do so if G is countable and every vertex in X has finite degree.

7) Show that $\kappa(G) \leq \lambda(G) \leq \delta(G)$.
   Conversely, show that if $1 \leq k \leq l \leq d$ are integers, then there is a graph with $\kappa(G) = k$, $\lambda(G) = l$ and $\delta(G) = d$.
   Construct a graph H with a vertex v such that $\kappa(H) = k$ and $\kappa(H - v) = l$.
   In the special case that G is *cubic* (all degrees are 3) prove that $\kappa(G) = \lambda(G)$.

8) Prove that if G is k-connected ($k \geq 2$) and $\{x_1, x_2, \dots, x_k\} \subset V(G)$ then there is a cycle in G of length at least $k+1$ that contains all x_i, $1 \leq i \leq k$.

Further Problems: the selection above covers the course but you might enjoy the ones below too.

F1) A *tournament* is a complete *oriented* graph, that is, a complete graph in which each edge uv is given a direction, either from u to v or from v to u. Prove that every tournament contains a (directed) path containing every vertex.

F2) Let G be a graph of order n, with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n = \Delta$, such that $d_k \geq k$ for $k \leq n - \Delta - 1$. Prove that G is connected.

F3) Show that every forest of order n contains either at least $n/9$ leaves or at least $n/9$ vertex disjoint paths of length 4.

F4) Why is the maximum number of edges in a graph having order n and having no odd cycles precisely $\lfloor n^2/4 \rfloor$?
Show that, in a graph containing no even cycles, every vertex of degree greater than two is a cutvertex (i.e., its removal disconnects the graph). Deduce that the maximum size of a graph of order n having only odd cycles is $\lfloor 3(n-1)/2 \rfloor$.

F5) Show that if G is a regular (all degrees equal) bipartite graph then $\kappa(G) \neq 1$.

F6) Let H be a subgroup of the group G having finite index k. Show that there are elements $g_1, \dots, g_k \in G$ such that $g_1H, \dots, g_kH$ are the distinct left cosets and $Hg_1, \dots, Hg_k$ are the distinct right cosets. (P. Hall)

F7) Let uv be an edge of the graph G. The graph G/uv is obtained by *contracting* the edge uv; that is, the edge uv is removed, u and v are identified and any resulting duplicate edges are deleted. H is a *minor* of G, written $H \prec G$, if it is a subgraph of a graph obtained from G by a sequence of edge-contractions.
 (i) Observe that $G \succ H$ if and only if $V(G)$ contains disjoint subsets W_v, $v \in V(H)$, such that $G[W_v]$ is connected and, whenever $uv \in E(H)$, there is an edge of G between W_u and W_v.
 (ii) Deduce that, if $\Delta(H) \leq 3$, then $G \succ H$ if and only if G contains a subdivision of H.
 (iii) Prove that G is planar if and only if $G \not\succ K_{3,3}$ and $G \not\succ K_5$.

F8) Refer to the previous exercise for the definition of $G \succ H$ and the fact that $G \succ K_4$ if and only if G contains a subdivision of K_4.
 (i) Show that if $\kappa(G) \geq 3$ then $G \succ K_4$.
 (ii) Show that if $G \not\succ K_4$ then G has at least two vertices of degree at most 2.
 (iii) Deduce that if $e(G) \geq 2|G| - 2$ then G contains a subdivision of K_4.

F9) Show that the *Petersen* graph (as shown) is non-planar by
 (a) showing it has too many edges
 (b) finding a subdivision of $K_{3,3}$
 (c) finding a K_5 minor.