MATHEMATICAL TRIPOS PART II (2004–05)

Graph Theory - Problem Sheet 3 of 4 **A.G. Thomason**

Note: The exercises are largely independent of each other — so, if you can't do one, go on to another.

27) Show that $e(G) \geq \binom{\chi(G)}{2}$ holds for every graph G.

28) Show that, for any graph G, there is an ordering of the vertices on which the greedy colouring algorithm uses only $\chi(G)$ colours.

29) Let $k \geq 3$ and $n = 2k-2$. Find a bipartite graph with vertices $v_1, v_2, \ldots, v_n$ for which the greedy colouring algorithm uses k colours. Is there such a graph with $n = 2k-3$?

30) Let G be a graph of order n, and let $\overline{G}$ be its complement. Show that $n \leq \chi(G)\chi(\overline{G})$ and deduce that $2\sqrt{n} \leq \chi(G) + \chi(\overline{G})$.

31) Show directly that $p_G(x)$ is a polynomial (i.e., without using induction) because $p_G(x) = \sum_{i=1}^{n} c_i x(x-1)\ldots(x-i+1)$, where c_i is the number of ways to partition G into i non-empty independent (that is, edge-free) sets.

32) Let $p_G(x) = \sum_{i=0}^{n}(-1)^i a_i x^{n-i}$ be the chromatic polynomial of G. Prove that $a_i \geq 0$, $a_0 = 1$, $a_1 = e(G)$ and $a_2 = \binom{e(G)}{2} - t(G)$, where $t(G)$ is the number of triangles in G.

33) Find graphs G and H with $|G| = |H|$, $e(G) = e(H)$ and $\chi(G) < \chi(H)$, and such that there are more ways to colour G than H when the number of available colours is large. Do the same again but with $\chi(G) > \chi(H)$.

34) Show that $|p_G(-1)| = (-1)^{|G|} p_G(-1)$ is the number of acyclic orientations of G. (An orientation of G is an assignment of a direction to each edge.)

35) Let $n = 2^p$. Show that K_{n+1} is not the union of p bipartite graphs but that K_n is. Deduce that among any $2^p + 1$ points in the plane there are three that determine an angle of size at least $\pi(1 - (1/p))$.

36) Suppose that G_k is a triangle-free graph with vertex set $\{u_1, \ldots, u_n\}$ and satisfying $\chi(G_k) = k$. Construct a graph G_{k+1} from G_k by adding new vertices $\{w, v_1, \ldots, v_n\}$ so that v_i is joined to $\Gamma_{G_k}(u_i) \cup \{w\}$. Show that G_{k+1} is triangle-free and that $\chi(G_{k+1}) = k + 1$. Construct explicitly such graphs for $k \leq 4$.

37) Let G be the graph of order 2n+1 obtained by subdividing a single edge of $K_{n,n}$ by a new vertex. Show that $\chi'(G) = \Delta(G) + 1$, but that if e is any edge of G then $\chi'(G - e) = \Delta(G - e)$.

38) By examining the proof of Vizing's theorem, show that a graph G with at most two vertices of degree $\Delta(G)$ satisfies $\chi'(G) = \Delta(G)$.

39) Show that a list colouring of an odd cycle is possible when every vertex has a list of two colours unless the lists are identical.

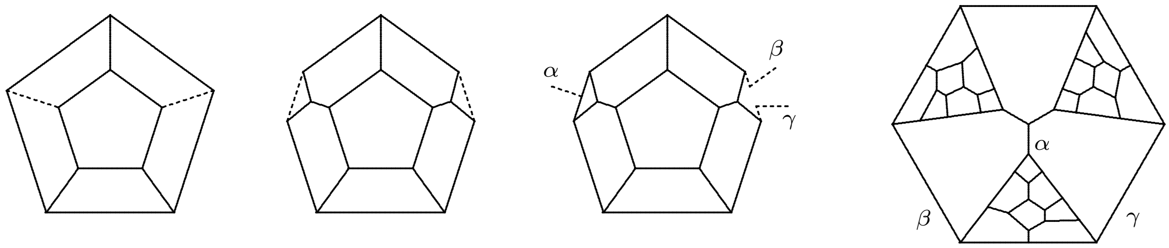
40) Let $G = K_{n,n}$, so $\chi(G) = 2$. If $n \geq \binom{2k-1}{k}$ show that $\chi_\ell(G) \geq k + 1$ by assigning lists of k colours to each vertex so that there is no list colouring.

41) Draw the maps of the five Platonic solids. What are the dual maps?

42) Show that an Eulerian plane map is 2-colourable.

43) Prove that a triangulated plane map is 3-colourable unless it is K_4.

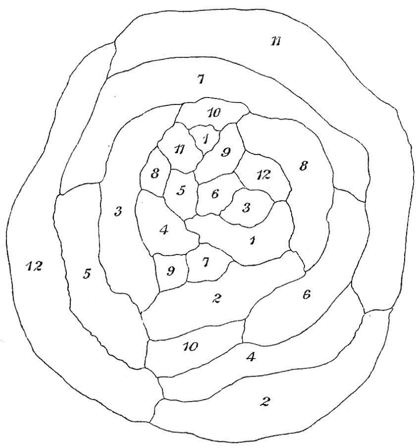
44) Verify Tutte's counterexample of order 46 to Tait's conjecture (the dashes might help).



45) Let S be the projective plane. Show that $\max\{\chi(G) : G \text{ embeds on } S\} = 6$.

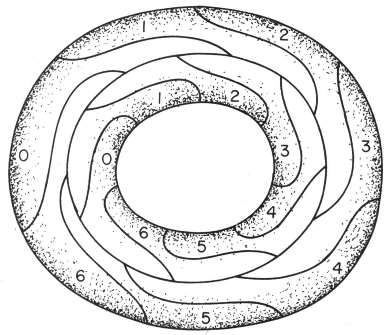
46) Show that the chromatic number of a triangle-free graph embedded on a surface of Euler characteristic $E \leq 0$ is at most $(5 + \sqrt{25 - 16E})/2$.

47) In the colouring of a plane map, an *m-pire* is a set of up to m faces that must receive the same colour (eg France and its overseas departments). Show that if a map has m-pires it can be coloured with $6m$ colours. Find a 2-pire map that needs 12 colours.



Heawood's 2-pire map

(and Heawood's hoop)



Further Problems

Note: the examples above are minimal to cover the course; you are encouraged to do those below also.

F15) Let $d_1 \geq d_2 \geq \ldots \geq d_n$ be the degree sequence of G. Show that if the vertices are taken in the order $x_1, x_2, \ldots, x_n$ where $d(x_i) = d_i$, then the greedy algorithm uses at most $\max_{1 \leq i \leq n} \min\{d_i + 1,\ i\}$ colours, and so if k is the maximum natural number for which $k \leq d_k + 1$ then $\chi(G) \leq k$. Deduce that $\chi(G) + \chi(\overline{G}) \leq n + 1$.

F16) Let G be a connected graph with blocks $B_1, B_2, \ldots, B_t$ (recall that a *block* is a maximal 2-connected subgraph). Show that $p_G(x) = x^{1-t} \prod_{i=1}^{t} p_{B_i}(x)$.

F17) Show that if G is connected then $(-1)^{|G|-1} p_G(x) > 0$ for all x, $0 < x < 1$.

F18) Let $u \in G$ and let $U_k = \{v \in G : d(u,v) = k\}$. Show that $\chi(G) \leq \chi(G[U_k]) + \chi(G[U_{k+1}])$ for some k. Deduce that for every natural number p there is a minimal integer $c(p)$ such that every graph G with chromatic number at least $c(p)$ has K_p as a minor (that is, $G \succ K_p$), and indeed $c(1) = 1$, $c(2) = 2$ and $c(p+1) \leq 2c(p) - 1$ for $p \geq 2$. Show further that $c(3) = 3$ and $c(4) = 4$. (The identity $c(5) = 5$ implies, and can be shown to be equivalent to, the four colour theorem.)