MATHEMATICAL TRIPOS PART II (2004–05)

Graph Theory - Problem Sheet 2 of 4 **A.G. Thomason**

Note: The exercises are largely independent of each other — so, if you can't do one, go on to another.

15) Let G be a graph of order n.
   (i) Let x and y be non-adjacent vertices with $d(x) + d(y) \geq n$. Show that G is Hamiltonian (has a Hamiltonian cycle) if and only if $G + xy$ is.
   (ii) Form the *closure* $C(G)$ of G by repeatedly joining pairs of vertices whose degree sum is at least n, until no such pairs remain. Show that $C(G)$ is well-defined (that is, the order in which pairs are joined is immaterial).
   (iii) Deduce that, if $d(x) + d(y) \geq n$ whenever $xy \notin E(G)$, then G is Hamiltonian.

16) Construct a graph of order n with no Hamiltonian cycle and with size $\binom{n}{2} - (n-2)$. Show that no greater size can be achieved.

17) Show directly, without appealing to Turán's theorem, that if G is a graph of order n and $\delta(G) > \delta(T_r(n)) = n - \lceil n/r \rceil$ then G contains K_{r+1}.

18) Let $v_1, \ldots, v_n$ be vectors of length at least one in some Euclidean space. Prove that there are at most $\lfloor n^2/4 \rfloor$ pairs $i < j$ with $||v_i + v_j|| < 1$.
[Hint. Show there's a pair with $||v_i + v_j|| \geq 1$ and $1 \leq i < j \leq 3$.]

19) Let $v_1, \ldots, v_{3n}$ be vectors in $\mathbb{R}^2$ with $||v_i - v_j|| \leq 1$ for all i, j. Prove that at most $3n^2$ of the distances $||v_i - v_j||$ can, in fact, exceed $1/\sqrt{2}$.
[Hint. Can all the distances amongst four of the points exceed $1/\sqrt{2}$?]

20) Let G have $n \geq r + 1$ vertices and $t_{r-1}(n) + 1$ edges.
   (i) Prove that for every p in the range $r \leq p \leq n$, G has a subgraph of order p and size at least $t_{r-1}(p) + 1$.
   (ii) Deduce that if $r \geq 3$ then G contains all but one edge of a K_{r+1}.

21) Prove that for $n \geq 5$ every graph of order n with $\lfloor n^2/4 \rfloor + 2$ edges contains two triangles with exactly one vertex in common.

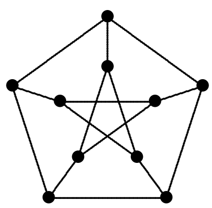
22) For each p in the range $2 \leq p \leq n/2$, construct a connected regular graph of order n containing K_p but not K_{p+1}. On the other hand, show that if $p > n/2$ and G is a regular graph of order n containing K_p, then $G = K_n$.

23) Prove that if $|G| = n$ and $e(G) > \dfrac{n}{4}\{1 + \sqrt{4n-3}\}$ then G contains a cycle of length 4.

24) Let G be a graph of order n and let $G_1, \ldots, G_n$ be the subgraphs of order $n - 1$ obtainable by removing a single vertex. Show that $\sum_{i=1}^{n} e(G_i) = (n-2)e(G)$.
Let F be a graph and let $c_n = \mathrm{ex}(n; F)/\binom{n}{2}$. Deduce that $c_n \leq c_{n-1}$ and hence that $\lim_{n\to\infty} c_n$ exists.

25) Show that $\lim_{n\to\infty} \mathrm{ex}(n; P)/\binom{n}{2} = \frac{1}{2}$, where P is the cubic graph of order 10 called the *Petersen* graph, shown here.

The Petersen Graph



26) The *upper density* ud(G) of an infinite graph G is the supremum of the densities of its large finite subgraphs; that is,

$$\mathrm{ud}(G) = \lim_{n\to\infty} \sup \{ e(H)/\binom{|H|}{2} : H \subset G,\, n \le |H| < \infty \} .$$

Show that, for every G, $\mathrm{ud}(G) \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1-\frac{1}{r}, \ldots, 1\}$.

Further Problems

Note: the examples above are minimal to cover the course; you are encouraged to do those below also.

F8) Let $A = (a_{ij})_1^n$ be an $n \times n$ doubly stochastic matrix, that is, its entries are non-negative and the rows and columns each sum to one. Show that A is in the convex hull of the set of $n \times n$ permutation matrices, i.e. there are permutation matrices $P_1, P_2, \ldots, P_m$ such that $A = \sum_1^m \lambda_i P_i$ and $\sum_1^m \lambda_i = 1$. [Let $a_{ij}^* = \lceil a_{ij} \rceil$, $A^* = (a_{ij}^*)_1^n$, and let $G = G_2(n,n)$ be the bipartite graph naturally associated with A^*. Show that G has a 1-factor and deduce that one can find a permutation matrix P and a real λ, $0 < \lambda \le 1$, such that $A - \lambda P = B = (b_{ij})_1^n$ satisfies $b_{ij} \ge 0$ and B has at least one more 0 entry than A.] How small can one choose m?

F9) Reduce Menger's Theorem to Hall's Theorem in the following way. Let G be a minimal counterexample to the assertion that there are always $\kappa(a,b)$ vertex disjoint paths between two non-adjacent vertices a and b. Show, as in lectures, that if $\kappa(a,b) = k$ and S is a k-cut then $S \subset \Gamma(a)$ or $S \subset \Gamma(b)$.

Therefore (by minimality) no vertex lies outside $\Gamma(a) \cup \Gamma(b)$, else it could be removed. Likewise (by minimality) $\Gamma(a) \cap \Gamma(b) = \emptyset$, for any vertex in this set can be removed. But then $G - \{a,b\}$ has bipartition $\Gamma(a)$, $\Gamma(b)$ and we can apply Hall's Theorem.

F10) The *independence number* $\beta(G)$ of a graph is the size of a largest independent vertex subset (spanning no edges). Show that if $\beta(G) \le \kappa(G)$ then G is Hamiltonian.
[Hint. If not, let C be a largest cycle in G and let $x \in V(G) - V(C)$. Find $\kappa(G)$ paths from x to C, and consider the vertices on C preceding the endvertices of these paths.]

+F11) Show that an $r-$regular graph of order $2r+1$ is Hamiltonian.

F12) Let G be a graph of order n and average degree $d = 2e(G)/n$. Let S be the sum $\sum_{uv\in E(G)} d(u) + d(v)$. Show that $nd^2 \le S = \sum_{u\in G} d(u)^2$. Hence deduce that, if $d(u) + d(v) \le 2D$ whenever $uv \in E(G)$, then $d \le D$.
In particular, show that if G satisfies the Ore degree condition, namely $d(u)+d(v) \ge n$ whenever $uv \notin E(G)$, then $e(G) \ge n^2/4$.

F13) Let G be a graph of order n satisfying the *anti*-Ore condition; that is, $d(u)+d(v) \ge n$ whenever $uv \in E(G)$. What is the minimum value of $e(G)$ if $\delta(G) \ge 1$? Can you find the minimum if $\delta(G) = d \le n/2$?

F14) By inspecting the given proof of the Erdős-Stone Theorem, prove that there is a function $n_2(r,\epsilon)$ such that every graph G of order $n > n_2$ with $\delta(G) \ge (1 - 1/r + \epsilon)n$ contains a $K_{r+1}(t)$ where $t = \lceil 2^{-r+1}(1/(r-1)!)\epsilon \log n \rceil$, and every graph G of order n with $e(G) \ge (1 - 1/r + \epsilon)\binom{n}{2}$ contains a $K_{r+1}(t)$ where $t = \lceil 2^{-r-1}(1/(r-1)!)\epsilon \log n \rceil$.