MATHEMATICAL TRIPOS PART II (2004–05)

Graph Theory - Problem Sheet 1 of 4 **A.G. Thomason**

Note: The exercises are largely independent of each other — so, if you can't do one, go on to another.

1) Show that every graph of order at least two contains two vertices of the same degree.

2) Let $d_1 \leq d_2 \leq \ldots \leq d_n$ be a sequence of integers. Show that there is a tree with degree sequence $(d_i)_1^n$ if and only if $d_1 \geq 1$ and $\sum_{i=1}^n d_i = 2n-2$. (Thus the conditions $d_i \geq 1$ and $\sum_{i=1}^n d_i = 2n-2$ characterize the degree sequences of trees.)

3) An *Euler trail* from a to b in a graph is a walk from a to b that uses every edge exactly once and each vertex at least once. Show that G has an Euler trail from a to b, $a \neq b$, if and only if G is connected and a and b are the only vertices of odd degree in G.

4) A *tournament* is a complete *oriented* graph, that is, a complete graph in which each edge uv is given a direction, either from u to v or from v to u. Prove that every tournament contains a (directed) path containing every vertex.

5) Let G be a graph. Show that its vertex set V has a partition $V = V_1 \cup V_2$ such that

$$e(G[V_1]) + e(G[V_2]) \leq \frac{1}{2}e(G).$$

Show also that one may also demand that each V_i span at most a third of the edges; that is, $e(G[V_i]) \leq \frac{1}{3}e(G), \quad i = 1, 2.$

6) Show that every maximal planar graph of order $n \geq 3$ has $3n - 6$ edges.

7) Prove that every planar graph has a drawing in the plane in which every edge is a straight line segment. [Hint. Apply induction on the order of maximal planar graphs by omitting a suitable vertex.]

8) Show that every graph can be drawn in $\mathbb{R}^3$ without crossing edges.

9) A graph is *k-regular* if every vertex has degree k. Show that a k-regular bipartite graph has a 1-factor (i.e., 1-regular spanning subgraph). Deduce that its edges can be coloured with k colours so that no two incident edges have like colours.

10) Let G be a bipartite graph with bipartition $X \cup Y$ having a matching from X into Y.
   (i) Prove that there is a vertex $x \in X$ such that for every edge xy there is a matching from X to Y that contains xy.
   (ii) Deduce that if $|X| = m$ and $d(x) = d$ for every $x \in X$ then G contains at least $d!$ matchings if $d \leq m$ and at least $d(d-1)\ldots(d-m+1)$ matchings if $d > m$.

11) Show that Hall's condition is not sufficient to ensure a matching in an infinite bipartite graph. Prove, however, that it does suffice if G is countable and every vertex in X has finite degree.

12) Show that $\kappa(G) \leq \lambda(G) \leq \delta(G)$.
Conversely, show that if $1 \leq k \leq l \leq d$ are integers, then there is a graph with $\kappa(G) = k$, $\lambda(G) = l$ and $\delta(G) = d$.
Construct a graph H with a vertex v such that $\kappa(H) = k$ and $\kappa(H - v) = l$.
In the special case that G is *cubic* (all degrees are 3) prove that $\kappa(G) = \lambda(G)$.

13) Prove that a graph G is k-connected iff $|G| \geq k+1$ and for any $U \subset V(G)$ with $|U| \geq k$ and for any vertex $x \notin U$, there are k paths from x to U, any pair of paths having only the vertex x in common.

14) Prove that if G is k-connected ($k \geq 2$) and $\{x_1, x_2, \ldots, x_k\} \subset V(G)$ then there is a cycle in G of length at least $k+1$ that contains all x_i, $1 \leq i \leq k$.

Further Problems

Note: the examples above are a selection that cover the course; you might enjoy the ones below too.

F1) Show that every forest of order n contains either at least $n/9$ leaves or at least $n/9$ vertex disjoint paths of length 4.

F2) Why is the maximum number of edges in a graph having order n and having no odd cycles precisely $\lfloor n^2/4 \rfloor$?
Show that, in a graph containing no even cycles, every vertex of degree greater than two is a cutvertex (i.e., its removal disconnects the graph). Deduce that the maximum size of a graph of order n having only odd cycles is $\lfloor 3(n-1)/2 \rfloor$.

F3) Let uv be an edge of the graph G. The graph G/uv is obtained by *contracting* the edge uv; that is, the edge uv is removed, u and v are identified and any resulting duplicate edges are deleted. H is a *minor* of G, written $H \prec G$, if it is a subgraph of a graph obtained from G by a sequence of edge-contractions.
   (i) By considering the subgraphs of G that are contracted to single vertices, observe that $G \succ H$ if and only if $V(G)$ contains disjoint subsets W_v, $v \in V(H)$, such that $G[W_v]$ is connected and, whenever $uv \in E(H)$, there is an edge of G between W_u and W_v.
   (ii) Deduce that, if $\Delta(H) \leq 3$, then $G \succ H$ if and only if G contains a subdivision of H.
   (iii) Prove that G is planar if and only if $G \not\succ K_{3,3}$ and $G \not\succ K_5$.

F4) A bipartite graph with bipartition $X \cup Y$ is (r,s)-*regular* if every vertex in X has degree r and every vertex in Y has degree s. Show that, if $r \geq s$, there is a matching of X into Y. Deduce that, if $k < n/2$, it is possible to find a $(k+1)$-subset $B \subset \{1, \ldots, n\}$ for each k-subset $A \subset \{1, \ldots, n\}$ so that $A \subset B$ and different As receive different Bs.

F5) Show that if G is a regular (all degrees equal) bipartite graph then $\kappa(G) \neq 1$.

F6) Let G be a graph of order n, with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n = \Delta$, such that $d_k \geq k$ for $k \leq n - \Delta - 1$. Prove that G is connected.

F7) Refer to an earlier exercise for the definition of $G \succ H$ and the fact that $G \succ K_4$ if and only if G contains a subdivision of K_4.
   (i) Show that if $\kappa(G) \geq 3$ then $G \succ K_4$.
   (ii) Show that if $G \not\succ K_4$ then G has at least two vertices of degree at most 2.
   (iii) Deduce that if $e(G) \geq 2|G| - 2$ then G contains a subdivision of K_4.