Geometry & Groups, Part II: 2008-9: Sheet 4

1. Construct a Schottky group (i.e. a subgroup of the Möbius group generated by Möbius maps which pair suitable disjoint disks) whose limit set is contained in the real line.

2. For a Schottky group generated by maps pairing disks with disjoint closures, the quotient $\mathbb{H}^3/G$ is a "handlebody" (the open region in space bound by a surface of some genus ≥ 1). What is the quotient when the disks in one pair become tangent to one another at their boundary?

3. Show that the maps $x \mapsto x/3$ and $x \mapsto 2/3+x/3$ have non-empty invariant sets other than the Cantor set.

4. (i) Find two similarities S_1, S_2 of $\mathbb{R}$ such that the unit interval $[0,1]$ is the unique non-empty compact invariant set for the S_i.
   (ii) Write the Cantor set C as the invariant set of a collection of *three* similarities of $\mathbb{R}$, and hence (re-)compute its Hausdorff dimension.

5. Let F be a finite subset of $\mathbb{R}^n$. Show that the zero-dimensional Hausdorff measure $\mathcal{H}^0(F)$ is the cardinality of F.

6. (i) Compute the Hausdorff dimension of the Sierpinski carpet, given by cutting a square into nine equal pieces, and removing the central one.

   (ii) Let $F = \{x \in \mathbb{R} \,|\, x = b_m b_{m-1} \ldots b_1 . a_1 a_2 \ldots \text{ with } b_i, a_j \neq 5\}$ be those points on the line which admit decimal expansions omitting the number 5. What is $\dim_H(F)$?

   (iii) Construct a fractal in the plane whose Hausdorff dimension is given by the positive real solution s to the equation $4(\frac{1}{4})^s + (\frac{1}{2})^s = 1$.

7. Show that for infinitely many (or even every) $s \in [0,2]$ there is a totally disconnected subset $F \subset \mathbb{R}^2$ for which $\dim_H(F) = s$.

8. (i) Suppose $F \subset \mathbb{R}^n$ is written as $F = \cup_{i\in\mathbb{Z}} F_i$. Show $\dim_H(F) = sup_i\{\dim_H(F_i)\}$.

   (ii) Deduce that if $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then for every subset $F \subset \mathbb{R}$, $\dim_H(f(F)) \leq \dim_H(F)$.

   (iii) Now consider $f : \mathbb{R} \to \mathbb{R}$ taking $x \mapsto x^2$. By considering the square-root function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, or otherwise, show that for every $F \subset \mathbb{R}$, $\dim_H(f(F)) = \dim_H(F)$.

9. Let (X,d) be a metric space and $\cdots \supset A_n \supset A_{n+1} \supset \cdots$ be a sequence of decreasing non-empty compact subsets of X. Prove that the intersection $\cap_k A_k$ is non-empty and compact. If the A_k were non-empty and open would the intersection necessarily be (i) non-empty (ii) open ?

10. Show that Hausdorff distance $d_{Haus}(A, B)$ defines a metric space structure on the set of compact subsets of a given metric space.

11. Give explicit examples of Kleinian groups realising 3 different values of Hausdorff dimension for their limit sets. Justify your answer!

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