GEOMETRY AND GROUPS – Example Sheet 4

TKC Michaelmas 2006

1. Find two similarities C_1, C_2 of $\mathbb{R}$ for which the unit interval is the unique non-empty, compact, invariant set.
2. What is the 0-dimensional Hausdorff measure $\mathcal{H}^0(F)$ of a subset F of $\mathbb{R}^N$?
3. The *Sierpiński carpet* is obtained by dividing the unit square into 9 equal squares, removing the central one, and then repeating the process indefinitely. Compute the Hausdorff dimension of the Sierpiński carpet. (You should show that the conditions of any theorem you use are satisfied.)
4. Write the Cantor set as the invariant set for a collection of 3 similarities and show that the formula for the Hausdorff dimension still gives the correct result.
5. A set is constructed from the unit interval as follows. From each closed interval remove the middle open interval of width $t < 1$ times the length of that interval. Then repeat this process infinitely often. Calculate the Hausdorff dimension of the resulting set. Show that there are Hausdorff subsets of $\mathbb{R}$ with every dimension between 0 and 1.
6. Show that there is a totally disconnected subset M of $\mathbb{R}^2$ that has Hausdorff dimension d for every $d \in [0, 2]$.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be the map $f : x \mapsto x^2$. Show that $\dim_{\mathcal{H}}(f(M)) = \dim_{\mathcal{H}}(M)$ for every subset M of $\mathbb{R}$.
8. Let (X_n) be independent Bernoulli random variables that each take the values $0, 2$ with probability $\frac{1}{2}$. Define
$$\xi(\omega) = \sum_{n=1}^{\infty} \frac{X_n(\omega)}{3^n} \ .$$
Show that ξ is a random variable that takes values in the Cantor set C. Show that, for any set $U \subset \mathbb{R}$, we have
$$\mathbb{P}(\xi \in U) \leqslant 2\,\mathrm{diam}(U)^d \qquad \text{where } d = \frac{\log 2}{\log 3} \ .$$
[*Hint: Consider sets U with* $3^{-(k+1)} \leqslant \mathrm{diam}(U) < 3^{-k}$.] Deduce that $\mathcal{H}^d(C) \geqslant \frac{1}{2}$.
9. Let $\boldsymbol{y} = (y_k)$ and $\boldsymbol{z} = (z_k)$ be sequences in $\mathbb{Z}_2^{\mathbb{N}}$. Show that
$$d(\boldsymbol{y}, \boldsymbol{z}) = \begin{cases} 0 & \text{when } \boldsymbol{y} = \boldsymbol{z}; \\ 2^{-n} & \text{when } n = \min\{k : y_k \neq z_k\} \end{cases}$$
is a metric on $\mathbb{Z}_2^{\mathbb{N}}$. Show that the map
$$\phi : \mathbb{Z}_2^{\mathbb{N}} \to C \ ; \quad (y_k) \mapsto \sum \frac{2y_k}{3^k}$$
is a homeomorphism from $\mathbb{Z}_2^{\mathbb{N}}$ onto the Cantor set C with this metric.
The self-similarities of the Cantor set induce self-similarities of $\mathbb{Z}_2^{\mathbb{N}}$. What are these?
10. A map $f : M \to N$ is α-Hölder continuous if there is a constant $C < \infty$ with
$$d(f(x), f(y)) \leqslant C d(x, y)^{\alpha} \qquad \text{for all } \ x, y \in M \ .$$
Show that, for such a map,
$$\dim_{\mathcal{H}} f(M) \leqslant \frac{1}{\alpha} \dim_{\mathcal{H}} M \ .$$
11. Are the circles $\Gamma_j(k)$ for a Schottky group G uniquely determined by G? In other words, can you find two different sets of circles that give the same Schottky group?
12. Show that every element of a Schottky group, formed from discs which have disjoint closures, is either the identity, hyperbolic or loxodromic. Prove that the limit set for such a group is totally disconnected.

13. Prove that matrices $A, B \in \mathrm{SL}(2, \mathbb{C})$ satisfy

$$\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\mathrm{tr}(B) \ .$$

Deduce that the trace of every element in the group generated by A and B is determined by the 3 numbers $\mathrm{tr}(A), \mathrm{tr}(B)$ and $\mathrm{tr}(AB)$.

14. Show that any two triples of mutually tangent discs are conjugate to one another by a Möbius transformation. Why does this show that the Apollonian gasket is unique up to conjugacy?

15. Give explicit examples of Kleinian groups realising three different values of Hausdorff dimension for their limit sets. Justify your answer.

Please send any comments or corrections to me at: t.k.carne@dpmms.cam.ac.uk .