Galois Theory: Example Sheet 1 of 4

1. Let $f \in K[X]$ be a non-zero polynomial, and let L/K be a field extension with $L = K(\alpha)$ for some $\alpha \in L$ with $f(\alpha) = 0$. Show that $[L : K] \leqslant \deg f$, and that equality holds if and only if f is irreducible over K.

2. Let L/K be a quadratic extension, that is, a field extension of degree 2. Show that if the characteristic of K is not 2 then $L = K(\alpha)$ for some $\alpha \in L$ with $\alpha^2 \in K$.

   Show that if the characteristic is 2 then either $L = K(\alpha)$ for some α with $\alpha^2 \in K$, or $L = K(\alpha)$ for some α with $\alpha^2 + \alpha \in K$.

3. Let $f(X) = X^3 + X^2 - 2X + 1 \in \mathbb{Q}[X]$. Use Gauss's lemma to show that f is irreducible. Suppose that α has minimal polynomial f over $\mathbb{Q}$, and let $\beta = \alpha^4$. Find a, b, $c \in \mathbb{Q}$ such that $\beta = a + b\alpha + c\alpha^2$. Do the same for $\beta = (1 - \alpha^2)^{-1}$.

4. Find the minimal polynomials over $\mathbb{Q}$ of the complex numbers:
$$\sqrt[5]{3}, \quad i + \sqrt{2}, \quad \sin(2\pi/5), \quad e^{i\pi/6} - \sqrt{3}.$$

5. (i) Let L/K be a finite extension of prime degree. Show that there is no intermediate extension $K \subsetneqq F \subsetneqq L$.

   (ii) Let α be such that $[K(\alpha) : K]$ is odd. Show that $K(\alpha) = K(\alpha^2)$.

6. Let L/K be a finite extension and $f \in K[X]$ an irreducible polynomial of degree $d > 1$. Show that if d and $[L : K]$ are coprime then f has no roots in L. With the same hypotheses, must f be irreducible in $L[X]$?

7. Suppose that L/K is an extension with $[L : K] = 3$. Show that for any $x \in L$ and $y \in L \setminus K$ we can find $p, q, r, s \in K$ such that
$$x = \frac{p + qy}{r + sy}.$$

   [Hint: Consider four appropriate elements of the 3-dimensional vector space L.]

8. (i) Let K be a field, and $f = g/h \in K(X)$ a non-constant rational function. Find a polynomial in $K(f)[T]$ which has X as a root.

   (ii) Let L be a subfield of $K(X)$ containing K. Show that either $K(X)/L$ is finite, or $L = K$. Deduce that the only elements of $K(X)$ which are algebraic over K are constants.

   (iii) Find explicit $\beta, \gamma \in \mathbb{C}$ such that $\mathbb{Q}(\beta, \gamma)/\mathbb{Q}$ is not a simple extension, i.e., cannot be written as $\mathbb{Q}(\alpha)$ for any α.

9. Let K and L be subfields of a field M such that M/K is finite. Denote by KL the set of all finite sums $\sum x_i y_i$ with $x_i \in K$ and $y_i \in L$. Show that KL is a subfield of M, and that
$$[KL : K] \leqslant [L : K \cap L].$$

10. Show that a regular 7-gon is not constructible by ruler and compass.

11. Find a splitting field $K/\mathbb{Q}$ for each of the following polynomials, and calculate $[K : \mathbb{Q}]$ in each case:

$$X^4 - 5X^2 + 6, \quad X^4 - 7, \quad X^8 - 1, \quad X^3 - 3X + 1, \quad X^4 + 4.$$

[Hint: We saw in lectures that $2\cos(2\pi/9)$ is a root of $X^3 - 3X + 1$.]

12. Let $f \in K[X]$ be a polynomial of degree n. Let L be a splitting field for f over K. Show that $[L : K] \leqslant n!$ and that if f is irreducible then $[L : K] \geqslant n$.

Further problems

13. Let R be a ring, and K a subring of R which is a field. Show that if R is an integral domain and $\dim_K R < \infty$ then R is a field. Show that the result fails without the assumption that R is a domain.

14. (i) Let α be algebraic over K. Show that there are only finitely many intermediate fields $K \subset F \subset K(\alpha)$. [Hint: Consider the minimal polynomial of α over F.]

    (ii) Show that if L/K is a finite extension of infinite fields for which there exist only finitely many intermediate subfields $K \subset F \subset L$, then $L = K(\alpha)$ for some $\alpha \in L$.

15. Let L/K be a field extension, and $\phi\colon L \to L$ a K-homomorphism. Show that if L/K is algebraic then ϕ is an isomorphism. Does this hold without the hypothesis L/K algebraic?

16. Let L/K be an extension, and $\alpha, \beta \in L$ transcendental over K. Show that α is algebraic over $K(\beta)$ if and only if β is algebraic over $K(\alpha)$. [The elements α and β are then said to be *algebraically dependent* over K.]

17. Show that for any $n > 1$ the polynomial $X^n + X + 3$ is irreducible over $\mathbb{Q}$.