

## Example sheet 1, Galois Theory, 2021

1. Find the greatest common divisors of the polynomials  $f = X^3 - 3$  and  $g = X^2 - 4$  in  $\mathbb{Q}[X]$  and in  $\mathbb{F}_5[X]$ , expressing them in the form  $af + bg$  for polynomials  $a, b$ .

2. (Quadratic extensions) Let  $L/K$  be an extension of degree 2. Show that if the characteristic of  $K$  is not 2, then  $L = K(\alpha)$  for some  $\alpha \in L$  with  $\alpha^2 \in K$ .

Show that if the characteristic is 2, then either  $L = K(\alpha)$  with  $\alpha^2 \in K$ , or  $L = K(\alpha)$  with  $\alpha^2 + \alpha \in K$ .

3. (i) Let  $L/K$  be a finite extension whose degree is prime. Show that there is no intermediate extension  $L \supsetneq K' \supsetneq K$ .

(ii) Let  $\alpha$  be algebraic over  $K$  of odd degree. Show that  $K(\alpha) = K(\alpha^2)$ .

4. Let  $L/K$  be an extension and  $\alpha, \beta \in L$ . Show that if  $\alpha + \beta$  and  $\alpha\beta$  are algebraic over  $K$ , then  $\alpha$  and  $\beta$  are also algebraic over  $K$ .

5. Let  $L = K(\alpha, \beta)$ , with  $\deg_K \alpha = m$ ,  $\deg_K \beta = n$ , and  $\gcd(m, n) = 1$ . Show that  $[L : K] = mn$ .

6. Find the minimal polynomials over  $\mathbb{Q}$  of the complex numbers  $\sqrt[5]{3}$ ,  $i + \sqrt{2}$ ,  $\sin(2\pi/5)$  and  $e^{\pi i/6} - \sqrt{3}$ .

7. Let  $\alpha$  have minimal polynomial  $X^3 + X^2 - 2X + 1$  over  $\mathbb{Q}$ . Express  $(1 - \alpha^2)^{-1}$  as a linear combination of 1,  $\alpha$  and  $\alpha^2$ . Justify the assertion that the cubic is irreducible over  $\mathbb{Q}$ .

8. Let  $L/K$  be a finite extension and  $f \in K[X]$  an irreducible polynomial of degree  $d > 1$ . Show that if  $d$  and  $[L : K]$  are coprime,  $f$  has no roots in  $L$ .

9. (i) Let  $K$  be a field, and  $r = p/q \in K(X)$  a non-constant rational function. Find a polynomial in  $K(r)[T]$  which has  $X$  as a root.

(ii) Let  $L$  be a subfield of  $K(X)$  containing  $K$ . Show that either  $K(X)/L$  is finite, or  $L = K$ . Deduce that the only elements of  $K(X)$  which are algebraic over  $K$  are constants.

10. Find a splitting field  $K/\mathbb{Q}$  for each of the following polynomials, and calculate  $[K : \mathbb{Q}]$  in each case:

$$X^4 - 5X^2 + 6, \quad X^4 - 7, \quad X^8 - 1, \quad X^3 - 2, \quad X^4 + 4.$$

11. Show that if  $L$  is a splitting field for a polynomial in  $K[X]$  of degree  $n$ , then  $[L : K] \leq n!$ .

12. Write down the minimum polynomial for  $\sqrt{2} + \sqrt{3}$  over  $\mathbb{Q}$ . Show that it is reducible mod  $p$ , for all primes  $p$ .

13. Show that an algebraic extension  $L/K$  of fields is finite if and only if it is *finitely generated*; i.e. iff  $L = K(\alpha_1, \dots, \alpha_n)$  for some  $\alpha_i \in L$ . Prove that the algebraic numbers (zeros of polynomials with rational coefficients) form a subfield of  $\mathbb{C}$  which is not finitely generated over  $\mathbb{Q}$ .

14. Let  $R$  be a ring, and  $K$  a subring of  $R$  which is a field. Show that if  $R$  is an integral domain and  $\dim_K R < \infty$  then  $R$  is a field. Show that the result fails without the assumption that  $R$  is a domain.

15. Let  $K$  and  $L$  be subfields of a field  $M$  such that  $M/K$  is finite. Denote by  $KL$  the set of all finite sums  $\sum x_i y_i$  with  $x_i \in K$  and  $y_i \in L$ . Show that  $KL$  is a subfield of  $M$ , and that

$$[KL : K] \leq [L : K \cap L].$$

16. Suppose that  $L/K$  is an extension with  $[L : K] = 3$ . Show that for any  $x \in L$  and  $y \in L - K$  we can find  $p, q, r, s \in K$  such that  $x = \frac{p + qy}{r + sy}$ .

[Hint: Consider four appropriate elements of the 3-dimensional vector space  $L$ .]

17. Let  $L/K$  be an extension, and  $\alpha, \beta \in L$  transcendental over  $K$ . Show that  $\alpha$  is algebraic over  $K(\beta)$  iff  $\beta$  is algebraic over  $K(\alpha)$ . [ $\alpha, \beta$  are then said to be **algebraically dependent**.]

18. Show that for any  $n > 1$  the polynomial  $X^n + X + 3$  is irreducible over  $\mathbb{Q}$ .