II Galois Theory   Michaelmas Term 2017

EXAMPLE SHEET 4

1. Prove that the Galois group G of the polynomial $t^6 + 3$ over the rationals is of order 6. By explicitly describing the elements of G, show that they have orders 1,2 and 3. Hence deduce that G is isomorphic to S_3. Why does it follow that $t^6 + 3$ is reducible over the finite field of p elements of all primes p?

2. Let $f(t)$ be an irreducible cubic polynomial over a field K of characteristic $\neq 2$. Let Δ be a square root of the discriminant of $f(t)$. Show that $f(t)$ remains irreducible over $K(\Delta)$.

3. Let $f(t)$ be an irreducible separable quartic and $g(t)$ be its resolvent cubic. Show that the discriminant of $f(t)$ and $g(t)$ are the same.

4. Find the Galois group of the polynomial $f(t) = t^4 + t + 1$ over the fields of two and three elements. Hence or otherwise determine the Galois group of $f(T)$ over the rationals.

5. Determine the Galois group of the polynomial $f(t0 = t^5 - 15t - 3$ over the rationals.

6. Show the the polynomial $f(t) = t^5 + 27t + 16$ has no rational roots. Show that the splitting field over the field of three elements is an extension of degree 4. Hence deduce that $f(t)$ is irreducible over the rationals. Prove that $f(t)$ has precisely two (non-multiple) roots over the finite filed of 7 elements. Find the Galois groups of $f(t)$ over the rationals.

7. Show that the Galois groups of $f(t) = t^5 + 20t^2 - 2$ over the rationals is S_5. Now let K be a finite extension of the rationals of primed degree at least 7. Show that the Galois group of $f(t)$ over K is also S_5.

8. Let K be the rationals. Show that $K(\sqrt{2 + \sqrt{2 + \sqrt{2}}})$ is a Galois extension of K and find its Galois group.

9. (i) Show the Galois group of $f(t) = t^5 - 4t + 2$ over the rationals K is S_5, and determine the Galois group over $K(i)$.

(ii) Find the Galois group of $f(t) = t^4 - 4t + 2$ over the rationals K and over $K(i)$.

10. Let G be the group of invertible $n \times n$ upper triangular matrices with entries in a finite field F. Show that G is soluble.

11. Let K_1 and K_2 ne algebraically closed fields of the same characteristic. Show that either K_1 is isomorphic to a subfield of K_2, or K_2 is isomorphic to a subfield of K_1.

12. Express $\Sigma_{i \neq j} X_i^3 X_j$ as a polynomial in the elementary symmetric polynomials.

13. Let $L = K(X_1, X_2, X_3, X_4)$ be the field of rational functions in four variables over a field K and let $M = K(s_1, s_2, s_3, s_4)$ where s_1, s_2, s_3, s_4 are the elementary symmetric polynomials in L. Show that $X_1X_3 + X_2X_4$ has a cubic minimal polynomial over M. Let G be the dihedral subgroup of S_4 generated by the permutations $\sigma_1 = (1234)$ and $\sigma_2 = (13)$. Show the the fixed field of G is $M(X_1X_3 + X_2X_4)$. Find the fixed field of the subgroup H generated by σ_1^2 and σ_2.

14. Show that for any $n > 1$ the polynomial $t^n + t + 3$ is irreducible over the rationals. Determine its Galois group for $n \leq 5$.

brookes@dpmms.cam.ac.uk