II Galois Theory   Michaelmas Term 2017

EXAMPLE SHEET 2

1. (i) Let K be a field of characteristic $p > 0$ such that every element of K is a p^{th} power. Show that any irreducible polynomial over K is separable.

(ii) Deduce that if F is a finite field then any irreducible polynomial over F is separable.

(iii) A field is said to be $perfect$ if every finite extension of it is separable. Show that any field of characteristic zero is perfect, and that a field of characteristic $p > 0$ is perfect if and only if every element is a p^{th} power.

2. (i) Let K be a field of characteristic $p > 0$ and let α be algebraic over K. Show that α is not separable over K if and only if $K(\alpha)$ is not equal to $K(\alpha^p)$, and that if this is the case then p divides $|K(\alpha) : K|$.

(ii) Deduce that if $K \leq L$ is a finite inseparable extension of fields of characterstic p then p divides $|L : K|$.

3. Let a and b be distinct rational numbers. Find a primitive element for the field extension obtained from the rationals by adjoining $\sqrt{a}$ and $\sqrt{b}$.

4. Let F be the field of p elements, and let $L = F(X, Y)$ be the field of rational functions in X and Y. Let K be the subfield $F(X^p, Y^p)$. Show that for any f in L one has f^p in K and deduce that $K \leq L$ is not a simple extension.

5. Let F be a finite field. By considering the multiplicative group of F, or otherwise, write down a non-constnat polynomial over F which does not have a root in F.

6. Let $K \leq M \leq L$ be field extensions. Show that $K \leq L$ is separable if and only if both $K \leq M$ and $M \leq L$ are separable.

7. (i) Let α be algebraic over a field K. Show that there is only a finite number of intermediate subfields $K \leq M \leq K(\alpha)$.

(ii) Show that if $K \leq L$ is a finite extension of infinite fields for which there exist only finitely many intermediate subfields $K \leq M \leq L$ then $L = K(\alpha)$ for some α in L.

8. (i) Show that if $K \leq M \leq L$ are finite field extensions then $\mathrm{Tr}_{L/K} : L \longrightarrow K$ is the composite of $\mathrm{Tr}_{L/M}$ and $\mathrm{Tr}_{M/K}$.

(ii) Let $K \leq M$ be a finite field extension which is not separable. Show that $\mathrm{Tr}_{M/K} : M \longrightarrow K$ is the zero map.

9. Let $K \leq L$ be a field extension and $\phi : L \longrightarrow L$ be a K-homomorphism. Show that if $K \leq L$ is algebraic then ϕ is an isomorphism. Does this hold without the hypothesis that $K \leq L$ is algebraic?

10. Suppose that M and L are fields and $\phi_1, \ldots, \phi_n$ are distinct embeddings of M into L. Prove that there do not exist elements $\lambda_1, \ldots, \lambda_n$ of L, not all zero, such that $\lambda_1\phi_1(x) + \ldots + \lambda_n\phi_n(x) = 0$ for all $x \in M$. Deduce that if $K \leq M$ is a finite field extension and $\phi_1, \ldots, \phi_n$ are distinct K-automorphisms of M then $n \leq |M : K|$.

11. (i) Find an example of a field extension $K \leq L$ which is normal but not separable.

(ii) Find finite field extensions $K \leq M \leq L$ such that $K \leq L$ and $M \leq L$ are normal but $K \leq M$ is not normal.

12. Let $K \leq M$ be a finite field extension. Suppose $M = K(\alpha_i, \ldots, \alpha_n)$ and the minimal polynomials of each α_i over K split over M. Show that the extension $K \leq M$ is normal.

13. Give an example of a field K of characteristic $p > 0$, and α and β of the same degree of K so that $K(\alpha)$ is not isomorphic to $K(\beta)$. Does such an example exist if K is a finite field? Justify your answer.

14. Show that the only field homomorphism from the reals to the reals is the identity map.

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