II Galois Theory   Michaelmas Term 2017

EXAMPLE SHEET 1

1. Let $K \leq L$ be a field extension of degree 2. Show that if the characteristic of $K \neq 2$ then $L = K(\alpha)$ for some $\alpha \in L$ with $\alpha^2 \in K$.

Show that if the characteristic is 2 then either $L = K(\alpha)$ for some α with $\alpha^2 \in K$, or $L = K(\alpha)$ for some α with $\alpha^2 + \alpha \in K$.

2. (i) Le $K \leq L$ be a finite extension of prime degree. Show that there is no intermediate extension $K < M < L$. (ii) Let α be such that $|K(\alpha) : K|$ is odd. Show that $K(\alpha) = K(\alpha^2)$

3. Find the minimal polynomial over the rationals of the following complex numbers: $(i\sqrt{3} - 1)/2$, $i + \sqrt{2}$, $\sin(2\pi/5), 2\cos(\pi/9)$.

4. Let $f(t) = t^3 + t^2 - 2t + 1$. Show that $f(t)$ is irreducible as a rational polynomial. Suppose that $f(t)$ is the minimal polynomial of α over the rationals. and let $\beta = \alpha^4$.Find rationals a, b, c such that $\beta = a + b\alpha + c\alpha^2$. Do the same for $\beta = (1 - \alpha^2)^{-1}$.

5. Let $K \leq L$ be a finite extension and $f(t) \in K[t]$ be an irreducible polynomial of degree $d > 1$. Show that if d and $|L : K|$ are coprime then $f(t)$ has no roots in L.

6. (i) Let K be a field and $Y = p(X)/q(X) \in K(X)$ be a non-constant rational function. Find a polynomial in variable t with coefficients in $K(Y)$ which has X as a root. (ii) Let L be a subfield of $K(X)$ containing K. Show that either $K(X)$ is a finite extension of L or $L = K$. Deduce that the only elements of $K(X)$ which are algebraic over K are constants.

7. Show that a regular 7-gon is not constructible by ruler and compasses.

8. Show that the angle $\pi/3$ cannot be trisected using ruler and compasses. Furthermore, show that the angle θ cannot be trisected using rule and compasses if $4t^3 - 3t - \cos\theta$ is irreducible over the field generated by the rationals and $\cos\theta$. Is the converse true?

9. Show that if $f(t)$ is an irreducible polynomial of degree 2 over the field K, and $L = K(\alpha)$ where $f(\alpha) = 0$, then L is a splitting field for $f(t)$ over K.

10. Find a splitting field K for each of the following rational polynomials, and calculate the degree of K over the rationals: $t^4 - 5t^2 + 6, t^8 - 1, t^8 - 2, t^4 + 4$.

11. Show that if L is a splitting field for a polynomial in $K[t]$ of degree n, then $|L : K| \leq n!$.

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