II Galois Theory Michaelmas Term 2016

EXAMPLE SHEET 2

1. (i) Let K be a field of characteristic $p > 0$ such that every element of K is a p^{th} power. Show that any irreducible polynomial over K is separable.

(ii) Deduce that if F is a finite field then any irreducible polynomial over F is separable.

(iii) A field is said to be $perfect$ if every finite extension of it is separable. Show that any field of characteristic zero is perfect, and that a field of characteristic $p > 0$ is perfect if and only if every element is a p^{th} power.

2. (i) Let K be a field of characteristic $p > 0$ and let α be algebraic over K. Show that α is not separable over K if and only if $K(\alpha)$ is not equal to $K(\alpha^p)$, and that if this is the case then p divides $|K(\alpha) : K|$.

(ii) Deduce that if $K \leq L$ is a finite inseparable extension of fields of characterstic p then p divides $|L : K|$.

3. Let a and b be distinct rational numbers. Find a primitive element for the field extension obtained from the rationals by adjoining $\sqrt{a}$ and $\sqrt{b}$.

4. Let F be the field of p elements, and let $L = F(X, Y)$ be the field of rational functions in X and Y. Let K be the subfield $F(X^p, Y^p)$. Show that for any f in L one has f^p in K and deduce that $K \leq L$ is not a simple extension.

5. Let F be a finite field. By considering the multiplicative group of F, or otherwise, write down a non-constnat polynomial over F which does not have a root in F. Deduce that F cannot be algebraically closed.

6. Let K_1 and K_2 ne algebraically closed fields of the same characteristic. Show that either K_1 is isomorphic to a subfield of K_2, or K_2 is isomorphic to a subfield of K_1.

7. (i) Let α be algebraic over a field K. Show that there is only a finite number of intermediate subfields $K \leq M \leq K(\alpha)$.

(ii) Show that if $K \leq L$ is a finite extension of infinite fields for which there exist only finitely many intermediate subfields $K \leq M \leq L$ then $L = K(\alpha)$ for some α in L.

8. Let $K \leq L$ be a field extension and $\phi : L \longrightarrow L$ be a K-homomorphism. Show that if $K \leq L$ is algebraic then ϕ is an isomorphism. Does this hold without the hypothesis that $K \leq L$ is algebraic?

9. (i) Find an example of a field extension $K \leq L$ which is normal but not separable.

(ii) Find finite field extensions $K \leq M \leq L$ such that $K \leq M$ nad $M \leq L$ are normal but $K \leq M$ is not normal.

10. Show that the only field homomorphism from the reals to the reals is the identity map.

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