Part II Galois theory (2014–2015) Example Sheet 2

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(1) Let K be a finite field. By considering the multiplicative group $K^{\times}$, or otherwise, write down a non-constant polynomial over K which does not have a root in K. Deduce that K cannot be algebraically closed.

(2) Let K be field and $\overline{K}$ its algebraic closure. Assume $K \subseteq L$ is a finite field extension. Show that L is K-isomorphic to some subfield of $\overline{K}$.

(3) Let K_1 and K_2 be algebraically closed fields of the same characteristic. Show that either K_1 is isomorphic to a subfield of K_2 or K_2 is isomorphic to a subfield of K_1.

(4) Find an example of a field extension $K \subseteq L$ which is normal but not separable.

(5) Let $K \subseteq L$ be a field extension with $[L : K] = 2$. Show that the extension is normal.

(6) Find finite field extensions $K \subseteq F \subseteq L$ such that $K \subseteq F$ and $F \subseteq L$ are normal but $K \subseteq L$ is not normal.

(7) Let L be the splitting field of $t^3 - 2$ over $\mathbb{Q}$. Find a subgroup of $\mathrm{Gal}(L/\mathbb{Q})$ which is not a normal subgroup.

(8) Let $K \subseteq L$ be a finite Galois extension, and F, M intermediate fields.
What is the subgroup of $\mathrm{Gal}(L/K)$ corresponding to the subfield $F \cap M$? Show that if there is a K-isomorphism $F \to M$, then the subgroups $\mathrm{Gal}(L/F)$ and $\mathrm{Gal}(L/M)$ are conjugate in $\mathrm{Gal}(L/K)$.

(9) Show that $\mathbb{Q} \subseteq L = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$ is a Galois extension and determine its Galois group. Write down all the subgroups of $\mathrm{Gal}(L/\mathbb{Q})$ and the corresponding subfields of L.

(10) Show that for any natural number n there exists a Galois extension $K \subseteq L$ with $\mathrm{Gal}(L/K)$ isomorphic to S_n, the symmetric group of degree n. Show that for any finite group G there exists a Galois extension whose Galois group is isomorphic to G. (Hint: to prove the first claim, consider the field $L = \mathbb{Q}(t_1, \dots, t_n)$ of rational functions in $t_1, \dots, t_n$, then consider an action of S_n on L, etc.)

(11) Let L be the splitting field of $t^5 - 4t + 2$ over $\mathbb{Q}$. Show that $\mathbb{Q} \subseteq L$ is a Galois extension with Galois group isomorphic to S_5.

(12) Let L be the splitting field of $t^4 + t^3 + 1$ over a field K. Compute the Galois group $\mathrm{Gal}(L/K)$ for each of the following cases: $K = \mathbb{F}_2$, $K = \mathbb{F}_3$, and $K = \mathbb{F}_4$.

(13) Let p be a prime number and $L = \mathbb{F}_p(t)$ be the field of rational functions in t. Let $a \in \mathbb{F}_p$ be a non-zero element, and let $\varphi \in \mathrm{Aut}_{\mathbb{F}_p}(L)$ be the automorphism determined by $\varphi(t) = at$. Determine the subgroup $G \leq \mathrm{Aut}_{\mathbb{F}_p}(L)$ generated by φ, and its fixed field L^G.

(14) Show that there is at least one irreducible polynomial $f \in \mathbb{F}_5[t]$ with $\deg f = 17$.

(15) Compute $\Phi_{12} \in \mathbb{Z}[t]$, the 12-th cyclotomic polynomial.

(16) Let $K \subseteq L$ be an extension of finite fields. Show that L is the n-th cyclotomic extension of K for some n.

(17) Let L be the 7-th cyclotomic extension of $\mathbb{Q}$. Find all the intermediate fields $\mathbb{Q} \subseteq F \subseteq L$ and write each one as $\mathbb{Q}(\alpha)$ for some α. Which one of these intermediate fields is Galois over $\mathbb{Q}$?

(18) Let $\Phi_n \in \mathbb{Z}[t]$ denote the n-th cyclotomic polynomial. Show that:

(i) If $n > 1$ is odd, then $\Phi_{2n}(t) = \Phi_n(-t)$.
(ii) If p is a prime dividing n, then $\Phi_{np}(t) = \Phi_n(t^p)$.
(iii) If p and q are distinct primes, then the non-zero coefficients of Φ_{pq} are alternately $+1$ and -1. ([Hint: First show that if $1/(1-t^p)(1-t^q)$ is expanded as a power series in t, then the coefficients of t^m with $m < pq$ are either 0 or 1.)
(iv) If n is not divisible by at least three distinct odd primes, then the coefficients of Φ_n are $1, 0$ or -1.
(v) Φ_{105} has at least one coefficient which is not $1, 0$ or -1.

(19) Let $\mu = \exp(2\pi i/n)$ where $i = \sqrt{-1}$, and let $L = \mathbb{Q}(\mu)$ be the n-th cyclotomic extension of $\mathbb{Q}$. Show that the isomorphism $\mathrm{Gal}(L/\mathbb{Q}) \to (\mathbb{Z}/\langle n \rangle)^\times$ sends the automorphism given by complex conjugation to the class of -1. Deduce that if $n \geq 3$, then $[L : L \cap \mathbb{R}] = 2$ and show that $L \cap \mathbb{R} = \mathbb{Q}(\mu + \mu^{-1}) = \mathbb{Q}(\cos 2\pi/n)$.