Part II Galois theory (2014–2015) Example Sheet 1

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(1) Find the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$.

(2) Let $K \subseteq L$ be a finite field extension such that $[L : K]$ is prime. Show that any intermediate field $K \subseteq F \subseteq L$ is equal to K or equal to L.

(3) Let $K \subseteq L$ be a field extension of degree 2. Show that if the characteristic of K is not 2, then $L = K(\alpha)$ for some $\alpha \in L$ with $\alpha^2 \in K$. Show that if the characteristic is 2, then either $L = K(\alpha)$ with $\alpha^2 \in K$, or $L = K(\alpha)$ with $\alpha^2 + \alpha \in K$.

(4) Let $K \subseteq L$ be a field extension and $\alpha \in L$ an element with $[K(\alpha) : K]$ an odd number. Show that $K(\alpha) = K(\alpha^2)$.

(5) Let $K \subseteq L$ be a field extension and $\alpha, \beta \in L$. Show that $\alpha + \beta$ and $\alpha\beta$ are algebraic over K if and only if α and β are algebraic over K.

(6) Let K be a field and $K(s)$ the field of rational functions in s over K, i.e. the fraction field of the polynomial ring $K[s]$. Determine all the elements of $K(s)$ which are algebraic over K.

(7) Let L be the set of all the numbers in $\mathbb{C}$ which are algebraic over $\mathbb{Q}$. Show that L is a subfield of $\mathbb{C}$ and that $[L : \mathbb{Q}]$ is infinite.

(8) Let $K \subseteq L$ be a field extension and $\varphi \colon L \to L$ a K-homomorphism. Show that φ is a K-isomorphism if L is algebraic over K.

(9) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Calculate $[L : \mathbb{Q}]$ and $\mathrm{Aut}_{\mathbb{Q}}(L)$. Is $\mathbb{Q} \subseteq L$ a Galois extension?

(10) Let $n \in \mathbb{N}$ and assume $f = t^{n-1} + t^{n-2} + \cdots + t + 1$ is irreducible in $\mathbb{Q}[t]$. Let $\mu = \exp(2\pi i/n)$ where $i = \sqrt{-1}$. Show that f is the minimal polynomial of μ over $\mathbb{Q}$. Next show that $\mathbb{Q} \subseteq \mathbb{Q}(\mu)$ is a Galois extension.

(11) We use the notation and assumptions of the previous problem. Show that there is a natural group isomorphism $\mathrm{Gal}(\mathbb{Q}(\mu)/\mathbb{Q}) \to G$ where G is the multiplicative group of the unit elements of the ring $\mathbb{Z}/\langle n \rangle$.

(12) Find a splitting field L over $\mathbb{Q}$ for each of the following polynomials, and then calculate $[L : \mathbb{Q}]$ in each case:

$$t^4 - 5t^2 + 6,\ t^8 - 1,\ t^3 - 2$$

(13) Let $K \subseteq L$ be a field extension and $f \in K[t]$ an irreducible polynomial of degree 2. Show that if f has a root in L, then L contains a splitting field of f over K.

(14) Let K be a field and $f \in K[t]$ a polynomial of degree n. Show that if L is a splitting field of f over K, then $[L : K] \leq n!$.

(15) Let K be a field of characteristic $p > 0$ such that every element of K is a p-th power, i.e. for each $a \in K$ there is $b \in K$ with $a = b^p$. Show that any polynomial in $K[t]$ is separable.

(16) Let K be a finite field. Show that every polynomial in $K[t]$ is separable.

(17) Let $K \subseteq L$ be an extension of fields of characteristic $p > 0$, and let $\alpha \in L$ be algebraic over K. Show that α is not separable over K if and only if $K(\alpha) \neq K(\alpha^p)$, and that if this is the case, then p divides $[K(\alpha) : K]$.

(18) Let $K \subseteq L$ be a finite extension of fields of characteristic $p > 0$ which is not separable. Show that p divides $[L : K]$.

(19) Let $K \subseteq L$ be a finite field extension. Show that there is a unique intermediate field $K \subseteq F \subseteq L$ such that $K \subseteq F$ is separable but $F \subseteq L$ is *purely inseparable*, i.e. no element $\alpha \in L \setminus F$ is separable over F. We call F the *separable closure* of K in L. Show that $|\operatorname{Hom}_F(L, E)| \leq 1$ for every extension $F \subseteq E$.

(20) Let $K \subseteq L$ be a finite field extension inside $\mathbb{C}$. Show that if $K \neq L$, then $|\operatorname{Hom}_K(L, \mathbb{C})| \geq 2$.