**Stochastic Financial Models** Michael Tehranchi
Example sheet 4 - Michaelmas 2025

**1.** Let $(X_n)_{n\geq 0}$ be a simple symmetric random walk starting at $X_0 = 0$, so $X_n = \xi_1 + \ldots + \xi_n$ where $(\xi_n)_{n\geq 1}$ are independent copies of $\xi$ with $\mathbb{P}(\xi = \pm 1) = \frac{1}{2}$. For $\delta > 0$, let

$$W^{(\delta)}_{(n+p)\delta} = \sqrt{\delta}(X_n + p\xi_{n+1})$$

for integer $n \geq 0$ and $0 \leq p < 1$. Confirm that
(a) $t \mapsto W^{(\delta)}_t$ is continuous,
(b) $W^{(\delta)}_t - W^{(\delta)}_s$ is independent of $(W^{(\delta)}_u)_{0\leq u\leq s-\delta}$ where $\delta \leq s \leq t$
(c) $W^{(\delta)}_t - W^{(\delta)}_s$ converges in distribution to $N(0, t-s)$ as $\delta \downarrow 0$. [It is a fact that if $Y_\delta \to Y$ in distribution and $Z_\delta \to 0$ in distribution, then $Y_\delta + Z_\delta \to Y$ in distribution.]

**2.** If $(W_t)_{t\geq 0}$ is a Brownian motion, show that the following processes are martingales:
(i) $W_t^2 - t$.
(ii) $W_t^3 - 3tW_t$.
(iii) $e^{\theta W_t - \frac{1}{2}\theta^2 t}$ for any $\theta \in \mathbb{R}$.

**3.** Let $T_a = \inf\{t \geq 0 : W_t = a\}$ be the first time that a Brownian motion hit level $a > 0$. Using a suitable martingale and the optional stopping theorem, show that the Laplace transform of $T_a$ is given by

$$\mathbb{E}[e^{-\lambda T_a}] = e^{-a\sqrt{2\lambda}}.$$

[It is a fact that the optional stopping theorem holds for continuous martingales.]
For the brave of heart: confirm this by integrating the density of $T_a$ as derived from the reflection principle.

**4.** Let $(W_t)_{t\geq 0}$ be a Brownian motion. Show that the random variable $\sup_{u\geq 1} \frac{W_u}{u}$ has the same distribution as $|W_1|$. [Hint: Recall the time inversion of Brownian motion.]

**5.** Given constants $c$ and $\theta$ such that $\theta + 2c \neq 0$, use the Cameron–Martin theorem and the reflection principle to show that

$$\begin{aligned}\mathbb{E}[e^{\theta \max_{0\leq s\leq t}(W_s + cs)}] =& \mathbb{E}[e^{\theta(W_t+ct)^+}] + \frac{\theta}{\theta+2c}\mathbb{E}[e^{(\theta+2c)(W_t-ct)^+} - 1] \\ =& \frac{2}{\theta+2c}\Big(c\ \Phi(-c\sqrt{t}) + (\theta+c)e^{\frac{1}{2}\theta(\theta+2c)t}\Phi\big((\theta+c)\sqrt{t}\big)\Big)\end{aligned}$$

[Hint: $\mathbb{E}[e^{\theta X}] = 1 + \int_0^\infty \theta e^{\theta x}\mathbb{P}(X \geq x)dx$ for non-negative $X$.]

**6.** Let $v^{(\delta)}(t,x) = \mathbb{E}[g(x + W^{(\delta)}_t)]$ where $W^{(\delta)}_t$ is defined in Problem 1. Show that

$$\frac{v^{(\delta)}(t+\delta, x) - v^{(\delta)}(t,x)}{\delta} = \frac{v^{(\delta)}(t, x+\sqrt{\delta}) - 2v^{(\delta)}(t,x) + v^{(\delta)}(t, x-\sqrt{\delta})}{2\delta}$$

**7.** Show that the following functions satisfy the backward heat equation $\partial_t u + \frac{1}{2}\partial_{xx} u = 0$.
(i) $u(t,x) = x^2 - t$.
(ii) $u(t,x) = x^3 - 3tx$.
(iii) $u(t,x) = e^{\theta x - \theta^2 t/2}$ for any $\theta \in \mathbb{R}$.

**8.** Let $v$ solve the heat equation

$$\partial_\tau v = \frac{1}{2}\partial_{xx} v$$

and let $V(t,s) = e^{-r(T-t)}v(\sigma^2(T-t), \log s + (r-\sigma^2/2)(T-t))$. Verify that $V$ solves the Black–Scholes PDE

$$\partial_t V + rs\partial_s V + \frac{1}{2}\sigma^2 s^2 \partial_{ss} V = rV$$

**9.** In the Black–Scholes model, find the time-$t$ prices of European contingent claims which pay at time $T$ the amounts: (a) $\sqrt{S_T}$, (b) $\log S_T$, (c) $\int_0^T S_u du$

In each case, how many shares should be held at time $t$ to replicate the payout?

**10.** Let $\mathrm{EC}(S_0, K, \sigma, r, T)$ denote the initial price in of a European call option with strike $K$, expiry $T$ on an asset with initial price $S_0$, in the Black–Scholes model volatility with $\sigma$ and interest rate is $r$. Let $\mathrm{EP}(S_0, K, \sigma, r, T)$ be the price of the European put with the same parameters. Verify the *put-call symmetry* formula

$$\mathrm{EP}(S_0, K, \sigma, r, T) = \mathrm{EC}(Ke^{-rT}, S_0 e^{rT}, \sigma, r, T)$$

**11.** Show that $\mathrm{EC}(S_0, K, \sigma, r, T)$ is strictly decreasing in the strike price $K$, and is strictly increasing in the initial stock price $S_0$, in the volatility $\sigma$, in the interest rate $r$ and in the expiry $T$. Furthermore, show that is strictly convex in both $S_0$ and $K$.

What are the corresponding statements for the Black–Scholes price of a European put option?

**12.** A European *lookback* call option entitles the holder to buy one share of stock at the expiry time $T$ at the lowest price reached by the stock during the life of the option. Thus, if it is purchased at time 0, at time $T$ it pays off the amount $S_T - \inf_{0\le u\le T} S_u$. In the Black-Scholes model show that the initial price of such an option is

$$\frac{S_0}{a}\left[(a+1)\Phi\left(\tfrac{1}{2}(a+1)\sigma\sqrt{T}\right) - e^{-rT}(a-1)\Phi\left(\tfrac{1}{2}(a-1)\sigma\sqrt{T}\right) - 1\right],$$

assuming $r > 0$, where $a = 2r/\sigma^2$. [Hint: use Problem 5]

**13.** Using the notation of Problem 10, show that the initial Black–Scholes price of a down-and-out call with strike $K$ and a barrier at $B$, where $B < \min\{S_0, K\}$, can be expressed as

$$\mathrm{EC}(S_0, K, \sigma, r, T) - (B/S_0)^{2r/\sigma^2 - 1}\mathrm{EC}(B^2/S_0, K, \sigma, r, T).$$