Stochastic Financial Models   Michael Tehranchi
Example sheet 1 - Michaelmas 2025

For random vectors X and Y, the notations $\mathrm{Cov}(X,Y) = \mathbb{E}(XY^\top) - \mathbb{E}(X)\mathbb{E}(Y)^\top$ and $\mathrm{Cov}(X) = \mathrm{Cov}(X,X)$ are used.

Throughout the sheet, consider a market with risk-free interest rate r and d risky assets with time-t prices S_t, where $\mathbb{E}(S_1) = \mu$ and $\mathrm{Cov}(S_1) = V$. Assume that V is positive definite. The time-t wealth of an investor is denoted X_t. Given a utility function U and initial wealth X_0, let $\pi(Y)$ denote the indifference price of a claim with time-1 payout Y.

A function is said to be *suitable* for a problem if it is behaved well enough for the formal calculation to be justified.

1. Given X_0 and a function F, an investor maximises $F(\mathbb{E}(X_1), \mathrm{Var}(X_1))$. Suppose that $F(\cdot, \sigma^2)$ is increasing for all σ^2 and $F(m, \cdot)$ is decreasing for all m. If there is an optimal portfolio of risky assets, show that it is mean-variance efficient.

2. Suppose that the random variable Z has zero mean and that the function U is suitable. Let
$$\psi(m, \sigma) = \mathbb{E}[U(m + \sigma Z)].$$
Show that
(a) if U is increasing, then $\psi(\cdot, \sigma)$ is increasing for all σ
(b) if U is concave, then ψ concave, that is
$$\psi(pm_0 + qm_1, p\sigma_0 + q\sigma_1) \geq p\,\psi(m_0, \sigma_0) + q\,\psi(m_1, \sigma_1)$$
for all $m_0, m_1, \sigma_0, \sigma_1$ and $0 \leq p = 1 - q \leq 1$.
(c) if U is concave, then $\psi(m, \cdot)$ is decreasing on $[0, \infty)$ for all m

3. Suppose that there is $d = 1$ risky asset. Given X_0, an investor maximises the expected utility $\mathbb{E}[U(X_1)]$ for a suitable increasing, concave utility function U. Show that the optimal number of shares θ has the same sign as $\mu - (1 + r)S_0$. Does this agree with your intuition?

4. Suppose S_1 is Gaussian. Given X_0, an investor maximises $\mathbb{E}[U(X_1)]$ for a suitable increasing, concave utility function U. Show that the optimal portfolio is mean-variance efficient. [Hint: consider Problems 1 and 2.]

5. Some useful facts about the Gaussian distribution for later reference.
(a) Suppose that $X \sim N(\mu, \sigma^2)$ and f is suitable. Show that
  (i) $\mathbb{E}[e^X f(X)] = e^{\mu + \frac{1}{2}\sigma^2}\mathbb{E}[f(X + \sigma^2)]$
  (ii) $\mathrm{Cov}(X, f(X)) = \sigma^2\mathbb{E}[f'(X)]$
(b) Now let X and Y be jointly Gaussian random vectors such that $\mathrm{Cov}(X)$ is positive definite. Show that
  (i) X and $Y - b^\top X$ are independent, where $b = \mathrm{Cov}(X)^{-1}\mathrm{Cov}(X, Y)$
  (ii) if X is scalar then $\mathrm{Cov}(Y, f(X)) = \mathrm{Cov}(Y, X)\mathbb{E}[f'(X)]$

6. Reconsider Problem 4. Show that the optimal portfolio of risky assets can be written as
$$\theta = \lambda\,V^{-1}[\mu - (1 + r)S_0]$$
where
$$\lambda = -\frac{\mathbb{E}[U'(X_1^*)]}{\mathbb{E}[U''(X_1^*)]}$$

and X_1^* is the optimised time-1 wealth. Compute λ in the case where $U(x) = -e^{-\gamma x}$ for a given risk aversion parameter $\gamma > 0$. [Hint: use Problem 5(b)(ii).]

7. Show that $\pi(a + b^\top S_1 + Y) = \frac{a}{1+r} + b^\top S_0 + \pi(Y)$ for any constants $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$.

8. Show that the function $t \mapsto \frac{\pi(tY)}{t}$ is decreasing.

9. Let $U(x) = -e^{-\gamma x}$ where $\gamma > 0$ is the constant coefficient of absolute risk aversion. Consider a contingent claim whose payout Y is *independent* of the risky asset prices S_1. Show that

$$\pi(Y) = -\frac{1}{(1+r)\gamma} \log \mathbb{E}(e^{-\gamma Y})$$

Simplify this formula when Y is Gaussian.

10. Let $U(x) = -e^{-\gamma x}$ where $\gamma > 0$ and let S_1 and Y be jointly Gaussian. Derive the formula

$$\pi(Y) = \frac{1}{1+r}\left(\mathbb{E}(Y) - \frac{\gamma}{2}\mathrm{Var}(Y)\right)$$
$$- \frac{1}{1+r}\mathrm{Cov}(Y, S_1)V^{-1}\left(\mu - (1+r)S_0 - \frac{\gamma}{2}\mathrm{Cov}(S_1, Y)\right)$$

[Hint: use Problems 5(b)(i), 7 and 9.]

11. In lectures, we have thought of the time-0 prices S_0 as given, and computed agents' optimal portfolios based on this. In this problem we will *derive* this initial price by asking that the market is in equilibrium, i.e. that net supply equals net demand.

Suppose there is a total of $n_i > 0$ shares of asset i, and let $n = (n_1, \ldots, n_d)^\top$. Suppose that there are K agents in the market, where each agent chooses a mean-variance efficient portfolio. Show that the equilibrium time-0 prices for the risky assets is of the form

$$S_0 = \frac{1}{1+r}(\mu - \Gamma V n)$$

for a positive scalar Γ.

12. Suppose that X has the $N(\mu, \sigma^2)$ distribution under a probability measure $\mathbb{P}$. Let $\mathbb{Q}$ be the equivalent probability measure such that $\frac{d\mathbb{Q}}{d\mathbb{P}} \propto e^X$. Show that the distribution of X under $\mathbb{Q}$ is $N(\mu + \sigma^2, \sigma^2)$. [Hint: use Problem 5(a)(i).]