Stochastic Financial Models Michael Tehranchi
Example sheet 4 - Michaelmas 2023

1. Let $(X_n)_{n\geq 0}$ be a simple symmetric random walk starting at $X_0 = 0$, so $X_n = \xi_1 + \ldots + \xi_n$ where $(\xi_n)_{n\geq 1}$ are independent copies of ξ with $\mathbb{P}(\xi = \pm 1) = \frac{1}{2}$. For $\delta > 0$, let

$$W^{(\delta)}_{(n+p)\delta} = \sqrt{\delta}(X_n + p\xi_{n+1})$$

for integer $n \geq 0$ and $0 \leq p < 1$. Confirm that
(a) $t \mapsto W^{(\delta)}_t$ is continuous,
(b) $W^{(\delta)}_t - W^{(\delta)}_s$ is independent of $(W^{(\delta)}_u)_{0\leq u\leq s-\delta}$ where $\delta \leq s \leq t$
(c) $W^{(\delta)}_t - W^{(\delta)}_s$ converges in distribution to $N(0, t-s)$ as $\delta \downarrow 0$. [It is a fact that if $Y_\delta \to Y$ in distribution and $Z_\delta \to 0$ in distribution, then $Y_\delta + Z_\delta \to Y$ in distribution.]

2. If $(W_t)_{t\geq 0}$ is a Brownian motion, show that the following processes are martingales:
(i) $W_t^2 - t$.
(ii) $W_t^3 - 3tW_t$.
(iii) $e^{\theta W_t - \theta^2 t/2}$ for any $\theta \in \mathbb{R}$.

3. Let $T_a = \inf\{t \geq 0 : W_t = a\}$ be the first time that a Brownian motion hit level $a > 0$. Using a suitable martingale and the optional stopping theorem, show that the Laplace transform of T_a is given by

$$\mathbb{E}[e^{-\lambda T_a}] = e^{-a\sqrt{2\lambda}}.$$

[It is a fact that the optional stopping theorem holds for continuous martingales.]

For the brave of heart: confirm this by integrating the density of T_a as derived from the reflection principle.

4. Let $(W_t)_{t\geq 0}$ be a Brownian motion. Show that the random variable $\sup_{u\geq 1} \frac{W_u}{u}$ has the same distribution as $|W_1|$. [Hint: Recall the time inversion of Brownian motion.]

5. Given constants c and θ such that $\theta + 2c \neq 0$, use the Cameron–Martin theorem and the reflection principle to show that

$$\mathbb{E}[e^{\theta \max_{0\leq s\leq t}(W_s + cs)}] = \frac{2}{\theta + 2c}\left(c\,\Phi(-c\sqrt{t}) + (\theta + c)e^{\frac{1}{2}\theta(\theta+2c)t}\Phi\big((\theta + c)\sqrt{t}\big)\right)$$

[Hint: $\mathbb{E}[e^{\theta X}] = 1 + \int_0^\infty \theta e^{\theta x}\mathbb{P}(X \geq x)dx$ for non-negative X.]

6. Let $v^{(\delta)}(t, x) = \mathbb{E}[g(x + W^{(\delta)}_t)]$ where $W^{(\delta)}_t$ is defined in Problem 1. Show that

$$\frac{v^{(\delta)}(t+\delta, x) - v^{(\delta)}(t, x)}{\delta} = \frac{v^{(\delta)}(t, x + \sqrt{\delta}) - 2v^{(\delta)}(t, x) + v^{(\delta)}(t, x - \sqrt{\delta})}{2\delta}$$

7. Show that the following functions satisfy the backward heat equation $\partial_t u + \frac{1}{2}\partial_{xx}u = 0$.
(i) $u(t, x) = x^2 - t$.
(ii) $u(t, x) = x^3 - 3tx$.
(iii) $u(t, x) = e^{\theta x - \theta^2 t/2}$ for any $\theta \in \mathbb{R}$.

8. Let v solve the heat equation

$$\partial_\tau v = \frac{1}{2}\partial_{xx}v$$

and let $V(t,s) = e^{-r(T-t)}v(\sigma^2(T-t), \log s + (r-\sigma^2/2)(T-t))$. Verify that V solves the Black–Scholes PDE

$$\partial_t V + rs\partial_s V + \frac{1}{2}\sigma^2 s^2 \partial_{ss} V = rV$$

9. In the Black–Scholes model, find the time-t prices of European contingent claims which pay at time T the amounts: (a) $\sqrt{S_T}$, (b) $\log S_T$, (c) $\int_0^T S_u du$

In each case, how many shares should be held at time t to replicate the payout?

10. Let $\mathrm{EC}(S_0, K, \sigma, r, T)$ denote the initial price in of a European call option with strike K, expiry T on an asset with initial price S_0, in the Black–Scholes model volatility with σ and interest rate is r. Let $\mathrm{EP}(S_0, K, \sigma, r, T)$ be the price of the European put with the same parameters. Verify the *put-call symmetry* formula

$$\mathrm{EP}(S_0, K, \sigma, r, T) = \mathrm{EC}(Ke^{-rT}, S_0 e^{rT}, \sigma, r, T)$$

11. Show that $\mathrm{EC}(S_0, K, \sigma, r, T)$ is strictly decreasing in the strike price K, and is strictly increasing in the initial stock price S_0, in the volatility σ, in the interest rate r and in the expiry T. Furthermore, show that is strictly convex in both S_0 and K.

What are the corresponding statements for the Black–Scholes price of a European put option?

12. A European *lookback* call option entitles the holder to buy one share of stock at the expiry time T at the lowest price reached by the stock during the life of the option. Thus, if it is purchased at time 0, at time T it pays off the amount $S_T - \inf_{0 \le u \le T} S_u$. In the Black-Scholes model show that the initial price of such an option is

$$\frac{S_0}{a}\left[(a+1)\Phi\left(\tfrac{1}{2}(a+1)\sigma\sqrt{T}\right) - e^{-rT}(a-1)\Phi\left(\tfrac{1}{2}(a-1)\sigma\sqrt{T}\right) - 1\right],$$

assuming $r > 0$, where $a = 2r/\sigma^2$. [Hint: use Problem 5]

13. Using the notation of Problem 11, show that the initial Black–Scholes price of a down-and-out call with strike K and a barrier at B, where $B < \min\{S_0, K\}$, can be expressed as

$$\mathrm{EC}(S_0, K, \sigma, r, T) - (B/S_0)^{2r/\sigma^2 - 1}\mathrm{EC}(B^2/S_0, K, \sigma, r, T).$$