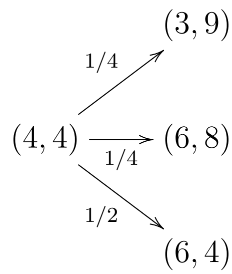
Stochastic Financial Models Michael Tehranchi

Example sheet 2 - Michaelmas 2023

1. Suppose the interest rate is $r = 1/4$, and there are $d = 2$ risky assets with prices



Find all arbitrage portfolios and/or all risk-neutral probability measures.

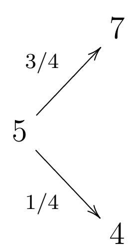
2. (Stiemke's theorem) Let A be a $m \times n$ matrix. Prove that exactly one of the following statements is true:

- There exists an $x \in \mathbb{R}^n$ with $x_i > 0$ for all $i = 1, \ldots, n$ such that $Ax = 0$.
- There exists a $y \in \mathbb{R}^m$ with $(A^T y)_i \geq 0$ for all $i = 1, \ldots, n$ such that $A^T y \neq 0$.

[Hint: relate the matrix A to a financial model, and use the fundamental theorem.]

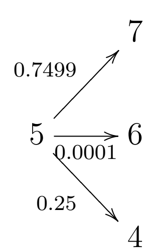
3. Consider a market with interest rate r and $d = 1$ risky assets. Given his initial wealth $X_0 = x$, an investor maximises the expected utility $\mathbb{E}[U(X_1)]$ for a suitable increasing, concave utility function U. Show that the optimal portfolio θ^* has the same sign as $\mathbb{E}(S_1) - (1+r)S_0$. Does this agree with your intuition?

4. Suppose the interest rate is $r = 1/3$, and there is $d = 1$ risky asset with prices



Introduce a call option with strike 6. Find its unique no-arbitrage initial price. Find a portfolio consisting of the risky and risk-free assets that replicates the payout of the call.

5. Consider the set-up of the previous problem, but now the prices are



Show directly that the payout of the call cannot be replicated by trading in the stock and bank account. Prove that there is no arbitrage if and only if the initial call price is in the open interval $(1/2, 2/3)$.

6. Consider a market with one risk-free asset with interest rate r and d risky assets with prices $(S_t)_{t\in\{0,1\}}$. Suppose this market has no arbitrage. Now introduce a claim with payout Y, and assume that there exists a unique price $\pi = y$ such that the augmented market also has no arbitrage. We will show that the claim is attainable in the original market.
(a) Why is there no loss to assume that the random variables $\{S_1^i - (1+r)S_0^i, 1 \le i \le d\}$ are linearly independent? Make this assumption for the rest of the problem.
(b) Why must there exist an equivalent measure $\mathbb{Q}$ such that $\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(S_1) = S_0$ and $\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(Y) = y$?
(c) Consider a market where the initial price of the claim is $\pi = y - \frac{1}{n}$. Show that there exists a portfolio $(\theta_n^\top, \varphi_n)^\top$ such that

$$\theta_n^\top[S_1 - (1+r)S_0] + \varphi_n[Y - (1+r)(y - \tfrac{1}{n})] \ge 0 \text{ almost surely.}$$

with strict inequality with positive probability.
(d) By computing the expected value of both sides of the inequality in (c) with respect to a risk-neutral measure $\mathbb{Q}$ for the original market, conclude that $\varphi_n > 0$. Explain why we may henceforth assume $\varphi_n = 1$ without loss of generality.
(e) Suppose the sequence $(\theta_n)_n$ is unbounded. By considering the sequence $(\lambda_n)_n$ where $\lambda_n = \theta_n/\|\theta_n\|$, show that there is a non-zero vector λ_0 such that

$$\lambda_0^\top[S_1 - (1+r)S_0] \ge 0 \text{ almost surely.}$$

Why is this a contradiction?
(f) Suppose the sequence $(\theta_n)_n$ is bounded. Show that there exists a portfolio θ_0 such that

$$Y \ge (1+r)y + \theta_0^\top[S_1 - (1+r)S_0] \text{ almost surely.}$$

Why does it follow that there is almost sure equality in the above inequality?

7. Let $(X_n)_{n\ge1}$ be an IID sequence, and let $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$. Let $\phi(t) = \mathbb{E}(e^{tX_n})$ and suppose ϕ is finite-valued, and that $\phi'(0) = \mathbb{E}(X_n) = 0$ and $\phi''(0) = \mathrm{Var}(X_n) = \sigma^2$.
(a) Show that the following processes are martingales, and specify the filtration.
  (i) $(S_n)_{n\ge0}$
  (ii) $(Q_n)_{n\ge0}$ where $Q_n = S_n^2 - \sigma^2 n$
  (iii) $(M_n)_{n\ge0}$ where $M_n = e^{tS_n}\phi(t)^{-n}$
(b) Show that

$$\mathbb{E}(X_1|S_n) = \frac{S_n}{n}.$$

8. Suppose $(M_n)_{n\ge0}$ is a martingale. Show that

$$\mathbb{E}(M_n|M_{n-1}) = M_{n-1} \text{ for all } n \ge 1$$

9. Find three random variables M_1, M_2, M_3 such that

$$\mathbb{E}(M_2|M_1) = M_1 \text{ and } \mathbb{E}(M_3|M_2) = M_2 \text{ yet } \mathbb{E}(M_3|M_1, M_2) \ne M_2$$

[Hint: consider problem 7(b).]

10. Consider a (homogenous) Markov-chain $(X_n)_{n\ge0}$ on a finite state-space S with transition matrix P. A function $f : S \to \mathbb{R}$ is considered as a column vector so that Pf makes sense as matrix multiplication. Let $\mathcal{F}_n = \sigma(X_k : 0 \le k \le n)$.
(a) Check that

$$\mathbb{E}[f(X_n)|\mathcal{F}_{n-1}] = [Pf](X_{n-1})$$

(b) Fix f and let

$$M_n = f(X_n) - \sum_{k=0}^{n-1} [(P - I)f](X_k).$$

Show that $(M_n)_{n \geq 0}$ is a martingale.

11. At time 1 an urn contains a white and a red ball. Take out a ball at random and replace it by two balls of the same colour; this gives the new content of the urn at time 2. Keep iterating this procedure.

Let W_n be the number of white balls in the urn at time n, and let $M_n = \frac{W_n}{n+1}$. Show that $(M_n)_{n \geq 1}$ is a martingale.