Stochastic Financial Models Michael Tehranchi

Example sheet 1 - Michaelmas 2023

For random vectors X and Y, the notation $\text{Cov}(X,Y) = \mathbb{E}(XY^\top) - \mathbb{E}(X)\mathbb{E}(Y)^\top$ and $\text{Cov}(X) = \text{Cov}(X,X)$ is used.

Throughout the sheet, consider a market with risk-free interest rate r and d risky assets with time-t prices S_t, where $\mathbb{E}(S_1) = \mu$ and $\text{Cov}(S_1) = V$. We assume that V is positive definite. The time-t wealth of an investor is denoted X_t. Given a utility function U and initial wealth X_0, let $\pi(Y)$ denote the indifference price of a claim with time-1 payout Y.

A function is said to be *suitable* for a problem if it is behaved well enough for the formal calculation to be justified.

1. Given X_0 and a function F, an investor maximises $F(\mathbb{E}(X_1), \text{Var}(X_1))$. Suppose that $F(\cdot, \sigma^2)$ is increasing for all σ^2 and $F(m, \cdot)$ is decreasing for all m. If there is a unique optimal portfolio of risky assets, show that it is mean-variance efficient.

2. Suppose that the random variable Z has zero mean and that the function U is suitable. Let

$$\psi(m, \sigma) = \mathbb{E}[U(m + \sigma Z)].$$

Show that

(a) if U is increasing, then $\psi(\cdot, \sigma)$ is increasing for all σ

(b) if U is concave, then ψ concave, that is

$$\psi(pm_0 + qm_1, p\sigma_0 + q\sigma_1) \geq p\ \psi(m_0, \sigma_0) + q\ \psi(m_1, \sigma_1)$$

for all $m_0, m_1, \sigma_0, \sigma_1$ and $0 \leq p = 1 - q \leq 1$.

(c) if U is concave, then $\psi(m, \cdot)$ is decreasing on $[0, \infty)$ for all m

3. Suppose S_1 is Gaussian. Given X_0, an investor maximises $\mathbb{E}[U(X_1)]$. Assume that the suitable function U is increasing and concave. Show that the optimal portfolio is mean-variance efficient. [Hint: consider Problems 1 and 2.]

4. Some useful facts about the Gaussian distribution for later reference.

(a) Suppose that $X \sim N(\mu, \sigma^2)$ and f is suitable. Show that

(i) $\mathbb{E}[e^X f(X)] = e^{\mu + \frac{1}{2}\sigma^2}\mathbb{E}[f(X + \sigma^2)]$

(ii) $\text{Cov}(X, f(X)) = \sigma^2 \mathbb{E}[f'(X)]$

(b) Now let X and Y be jointly Gaussian random vectors such that $\text{Cov}(X)$ is positive definite. Show that

(i) X and $Y - b^\top X$ are independent, where $b = \text{Cov}(X)^{-1}\text{Cov}(X, Y)$

(ii) if X is scalar then $\text{Cov}(Y, f(X)) = \text{Cov}(Y, X)\mathbb{E}[f'(X)]$

5. Reconsider Problem 3. Show that the optimal portfolio of risky assets can be written as

$$\theta^* = \lambda\ V^{-1}[\mu - (1 + r)S_0]$$

where

$$\lambda = -\frac{\mathbb{E}[U'(X_1^*)]}{\mathbb{E}[U''(X_1^*)]}$$

and X_1^* is the optimised time-1 wealth. Compute λ in the case where $U(x) = -e^{-\gamma x}$ for a given risk aversion parameter $\gamma > 0$. [Hint: use Problem 4(b).]

6. Suppose that X has the $N(\mu, \sigma^2)$ distribution under a probability measure $\mathbb{P}$. Let $\mathbb{Q}$ be the equivalent probability measure such that $\frac{d\mathbb{Q}}{d\mathbb{P}} \propto e^X$. Show that the distribution of X under $\mathbb{Q}$ is $N(\mu + \sigma^2, \sigma^2)$. [Hint: use Problem 4(a).]

7. Show that $\pi(a + b^\top S_1 + Y) = \frac{a}{1+r} + b^\top S_0 + \pi(Y)$ for any constants $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$.

8. Show that the function $t \mapsto \frac{\pi(tY)}{t}$ is decreasing.

9. Let $U(x) = -e^{-\gamma x}$ where $\gamma > 0$ is the constant coefficient of absolute risk aversion. Consider a contingent claim whose payout Y is *independent* of the risky asset prices S_1.
(a) Show that

$$\pi(Y) = \frac{1}{1+r} U^{-1} \left(\mathbb{E}[U(Y)] \right)$$

Simplify this formula when Y is Gaussian.
(b) Using the formula from part (a), verify that $\frac{1}{t}\pi(tY)$ is decreasing and that

$$\lim_{t \downarrow 0} \frac{1}{t} \pi(tY) = \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}(Y)$$

where $\mathbb{Q}$ is the risk-neutral probability measure with density $\frac{d\mathbb{Q}}{d\mathbb{P}} \propto U'(X_1^*)$ where X_1^* is the optimised time-1 wealth.

10. Let $U(x) = -e^{-\gamma x}$ where $\gamma > 0$ and let S_1 and Y be jointly Gaussian. Derive the formula

$$\pi(Y) = b^\top S_0 + \frac{1}{1+r} \left(\mathbb{E}(Y - b^\top S_1) - \frac{\gamma}{2} \mathrm{Var}(Y - b^\top S_1) \right)$$

where $b = V^{-1} \mathrm{Cov}(S_1, Y)$. [Hint: use Problems 4(b), 7 and 9(a).]

11. In lectures, we have thought of the time-0 prices S_0 as given, and computed agents' optimal portfolios based on this. In this problem we will *derive* this initial price by asking that the market is in equilibrium, i.e. that net supply equals net demand.

Suppose there is a total of $n_i > 0$ shares of asset i, and let $n = (n_1, \ldots, n_d)^\top$. Suppose that there are K agents in the market, where each agent chooses a mean-variance efficient portfolio. Show that the equilibrium time-0 prices for the risky assets is of the form

$$S_0 = \frac{1}{1+r} (\mu - \Gamma V n)$$

for a positive scalar Γ.