Stochastic Financial Models Michael Tehranchi

Example sheet 1 - Michaelmas 2022

Throughout this sheet, consider a market with risk-free interest rate r and d risky assets with initial prices S_0 and time-1 price S_1, where $\mathbb{E}(S_1) = \mu$ and $\text{Cov}(S_1) = V$. We assume that V is positive definite. Given a utility function U and initial wealth x, let $\pi(Y)$ denote the indifference price of a claim with time-1 payout Y.

Also, a function is said to be *suitable* for a problem if it is behaved well enough for the formal calculation to be justified.

1. Given an initial wealth $X_0 = x$, an investor tries to maximise $F(\mathbb{E}(X_1), \text{Var}(X_1))$ where X_1 is her wealth at time 1 and where the function F is given. Suppose that $F(\cdot, \sigma^2)$ is increasing for all σ^2 and $F(m, \cdot)$ is decreasing for all m. If there is a unique optimal portfolio of risky assets, show that it is mean-variance efficient.

2. Suppose that the random variable Z has zero mean and that the function U is suitable. Let

$$\psi(m, \sigma) = \mathbb{E}[U(m + \sigma Z)].$$

Show that

(a) if U is increasing, then $\psi(\cdot, \sigma)$ is increasing for all σ

(b) if U is concave, then ψ concave, that is

$$\psi(pm_0 + qm_1, p\sigma_0 + q\sigma_1) \geq p\psi(m_0, \sigma_0) + q\psi(m_1, \sigma_1)$$

for all $m_0, m_1, \sigma_0, \sigma_1$ and $0 \leq p = 1 - q \leq 1$.

(c) if U is concave, then $\psi(m, \cdot)$ is decreasing on $[0, \infty)$ for all m

3. Suppose S_1 is Gaussian. Given an initial wealth $X_0 = x$, an investor maximises $\mathbb{E}[U(X_1)]$ where X_1 is her wealth at time 1. Assume that the suitable function U is increasing and concave. Show that the optimal portfolio is mean-variance efficient.

4. Some useful facts about the Gaussian distribution for later reference.

(a) Suppose that $X \sim N(\mu, \sigma^2)$ and f is suitable. Show that

(i) $\mathbb{E}[e^X f(X)] = e^{\mu + \frac{1}{2}\sigma^2}\mathbb{E}[f(X + \sigma^2)]$

(ii) $\text{Cov}(X, f(X)) = \sigma^2 \mathbb{E}[f'(X)]$

(b) Now let X and Y be jointly Gaussian random vectors such that $\text{Cov}(X)$ is positive definite. Show that

(i) X and Z are independent, where $Z = Y - \text{Cov}(Y, X)\text{Cov}(X)^{-1}X$

(ii) $\text{Cov}(f(X), Y) = \mathbb{E}[f'(X)]\text{Cov}(X, Y)$ when X is scalar

5. Reconsider Problem 3. Show that the optimal portfolio of risky assets is

$$\theta^* = \lambda\, V^{-1}[\mu - (1 + r)S_0]$$

where

$$\lambda = -\frac{\mathbb{E}[U'(X_1^*)]}{\mathbb{E}[U''(X_1^*)]}$$

and $X_1^* = (1 + r)x + \theta^*[S_1 - (1 + r)S_0]$ is the optimised time-1 wealth. Compute λ in the case where $U(x) = -e^{-\gamma x}$ for a given risk aversion parameter $\gamma > 0$.

6. Suppose that X has the $N(\mu, \sigma^2)$ distribution under a probability measure $\mathbb{Q}$. Let $\mathbb{Q}$ be the equivalent probability measure such that $\frac{d\mathbb{Q}}{d\mathbb{P}} \propto e^X$. Show that the distribution of X under $\mathbb{Q}$ is $N(\mu + \sigma^2, \sigma^2)$.

7. Show that $\pi(a + b^\top S_1 + Y) = \frac{a}{1+r} + b^\top S_0 + \pi(Y)$ for any constants $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$.

8. Show that the function $t \mapsto \frac{\pi(tY)}{t}$ is decreasing.

9. Let $U(x) = -e^{-\gamma x}$ where $\gamma > 0$ is the constant coefficient of absolute risk aversion. Consider contingent claims whose payouts Y are independent of the risky asset prices S_1.

(a) Show that

$$\pi(Y) = \frac{1}{1+r} U^{-1}\left(\mathbb{E}[U(Y)]\right)$$

Simplify this formula when Y is Gaussian.

(b) Verify that the formula from part (a) is a concave function of Y.

(c) Verify that

$$\lim_{t \downarrow 0} \frac{1}{t}\pi(tY) = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(Y)$$

where $\mathbb{Q}$ is the risk-neutral probability measure whose density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is proportional to the marginal utility of optimised time-1 wealth $U'(X_1^*)$.

10. Let $U(x) = -e^{-\gamma x}$ where $\gamma > 0$ and let S_1 and Y be jointly Gaussian. Derive the formula

$$\pi(Y) = \theta^\top S_0 + \frac{1}{1+r}\Big(\mathbb{E}(Z) - \frac{\gamma}{2}\mathrm{Var}(Z)\Big)$$

where $b = V^{-1}\mathrm{Cov}(S_1, Y)$ and $Z = Y - b^\top S_1$.

11. In lectures, we have thought of the time-0 prices S_0 as given, and computed agents' optimal portfolios based on this. In this problem we will *derive* this initial price by asking that the market is in equilibrium, i.e. that net supply equals net demand.

Suppose there is a total of $n_i > 0$ shares of asset i, and let $n = (n_1, \ldots, n_d)^\top$. Suppose that there are K agents in the market, where each agent chooses a mean-variance efficient portfolio. Show that the equilibrium time-0 prices for the risky assets is of the form

$$S_0 = \frac{1}{1+r}(\mu - \Gamma V n)$$

for a positive scalar Γ.