Stochastic Financial Models Michael Tehranchi

Example sheet 1 - Michaelmas 2021

Throughout this sheet, we consider a market with risk-free interest rate r and d risky assets with initial prices S_0 and time-1 price S_1, where $\mathbb{E}(S_1) = \mu$ and $\mathrm{Cov}(S_1) = V$. We assume that V is positive definite. We will let $\pi(Y)$ denote the indifference price of a claim with time-1 payout Y.

Also, we say a function is *suitable* for a problem if it is behaved well enough for the formal calculation to be justified.

1. Given an initial wealth $X_0 = x$, an investor maximises $\mathbb{E}(X_1) - \frac{1}{2}\gamma \mathrm{Var}(X_1)$, where X_1 is her wealth at time 1, and $\gamma > 0$ is a risk-aversion parameter. Show that the optimal portfolio is mean-variance efficient.

2. An agent has expected utility preferences. Suppose that the agent is indifferent between X and Y iff $\mathbb{E}(X) = \mathbb{E}(Y)$ and $\mathrm{Var}(X) = \mathrm{Var}(Y)$. By considering distributions concentrated on three points, or otherwise, prove that the agent's utility function must be quadratic.

3. Suppose that the random variable Z has zero mean and that the function U is suitable. Let

$$\psi(m, \sigma) = \mathbb{E}[U(m + \sigma Z)].$$

Show that

(a) if U is increasing, then $\psi(\cdot, t)$ is increasing for all σ

(b) if U is concave, then ψ concave, that is

$$\psi(pm_0 + qm_1, p\sigma_0 + q\sigma_1) \geq p\psi(m_0, \sigma_0) + q\psi(m_1, \sigma_1)$$

for all $m_0, m_1, \sigma_0, \sigma_1$ and $0 \leq p = 1 - q \leq 1$.

(c) if U is concave, then $\psi(m, \cdot)$ is decreasing on $[0, \infty)$ for all m

4. Suppose S_1 is Gaussian. Given an initial wealth $X_0 = x$, an investor maximises $\mathbb{E}[U(X_1)]$ where X_1 is her wealth at time 1. Assume that the function U is concave and suitable. Show that the optimal portfolio is mean-variance efficient.

5. Some useful facts about the Gaussian distribution for later reference.

(a) Suppose that X is Gaussian. Show that
  (i) $\mathbb{E}[e^X f(X)] = e^{\mathbb{E}(X) + \frac{1}{2}\mathrm{Var}(X)} \mathbb{E}[f(X + \mathrm{Var}(X))]$
  (ii) $\mathrm{Cov}(X, f(X)) = \mathrm{Var}(X)\mathbb{E}[f'(X)]$ for suitable f,

(b) Suppose (X, Y) are jointly Gaussian and $\mathrm{Var}(X) > 0$ is positive definite. Show that
  (i) X and Z are independent, where $Z = Y - \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)} X$
  (ii) $\mathrm{Cov}(Y, f(X)) = \mathrm{Cov}(X, Y)\mathbb{E}[f'(X)]$ for suitable f.
  (ii) Let $\mathbb{Q}$ be the equivalent probability measure such that $\frac{d\mathbb{Q}}{d\mathbb{P}} \propto e^X$. Show that $\mathbb{E}^{\mathbb{Q}}(Y) = \mathbb{E}^{\mathbb{P}}(Y) + \mathrm{Cov}^{\mathbb{P}}(X, Y)$.

6. Reconsider Problem 4. Show that the optimal portfolio of risky assets is

$$\theta^* = \frac{1}{\gamma} V^{-1}[\mu - (1 + r)S_0]$$

where

$$\gamma = -\frac{\mathbb{E}[U''(X_1^*)]}{\mathbb{E}[U'(X_1^*)]}$$

and $X_1^* = (1+r)x + \theta^*[S_1 - (1+r)S_0]$ is the optimised time-1 wealth.

7. A contingent claim with time-1 payout Y is called *attainable* iff there exists a scalar a and portfolio $b \in \mathbb{R}^d$ such that $Y = a + b^\top S_1$.

(a) Show that the indifference price of an attainable contingent claim does not depend on the investor's utility function U or her initial wealth $X_0 = x$.

(b) Show that $\pi(Y+Z) = \pi(Y) + \pi(Z)$ whenever Y is the payout of an attainable claim.

8. Show that the function $t \mapsto \frac{\pi(tY)}{t}$ is decreasing.

9. Let S_1 and Y be jointly Gaussian and let $U(x) = -e^{-\gamma x}$ where $\gamma > 0$ is the constant coefficient of absolute risk aversion. Suppose Y is the payout of a contingent claim.

(a) What is the indifference price of this claim? [Hint: first consider the case where Y and S_1 are independent. For the general case, show that there exists a portfolio $\theta \in \mathbb{R}^d$ such that $Y = \theta^\top S_1 + Z$ where Z and S_1 are independent.]

(b) Verify that

$$\lim_{t \downarrow 0} \frac{1}{t}\pi(tY) = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(Y)$$

where $\mathbb{Q}$ be the risk-neutral probability measure whose density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is proportional to the marginal utility of optimised time-1 wealth $U'(X_1^*)$.

10. In lectures, we have thought of the time-0 prices S_0 as given, and computed agents' optimal portfolios based on this. In this problem we will *derive* this initial price by asking that the market is in equilibrium, i.e. that net supply equals net demand.

Suppose there is a total of n_i shares of asset i, and let $n = (n_1, \dots, n_d)^\top$. Suppose that there are K agents in the market, agent k solving the maximisation problem from Problem 1 above with a risk aversion γ_k. Show that the equilibrium time-0 prices for the risky assets must be

$$S_0 = \frac{(\mu - \Gamma V n)}{1+r}$$

where $\Gamma = (\sum_k \gamma_k^{-1})^{-1}$.

11. Consider Problem 10 above, but now suppose now that the agents decide to open a market in a contingent claim which is in zero net supply and with time-1 payout Y. Suppose the covariance matrix of the random vector $(S_1^\top, Y)^\top$ is positive definite.

(a) Show that the initial prices S_0 of the original assets are the same as in problem 10.

(b) Find the equilibrium price for the claim. Compare your answer to the indifference price calculated in Problem 9.

(c) Show that in equilibrium none of the agents hold the contingent claim. (This is surprising, as in the real world, contingent claims are very actively traded. What are some unrealistic assumptions of this model?)