Stochastic Financial Models Michael Tehranchi
Example sheet 3 - Michaelmas 2020

1. Given a filtration $(\mathcal{F}_n)_{n\geq 0}$ and a stopping time T, show that there exists an adapted process $(X_n)_{n\geq 0}$ such that $T = \inf\{n \geq 0 : X_n > 0\}$.

2. Let $(X_n)_{n\geq 0}$ be an integrable adapted process such that $\mathbb{E}(X_T) = X_0$ for all bounded stopping times T. Show that $(X_n)_{n\geq 0}$ is a martingale. Hint: Fix n and $A \in \mathcal{F}_{n-1}$ and consider the random time $T = n - \mathbb{1}_A$.

3. Let $(X_n)_{n\geq 0}$ be a square integrable martingale. Show that $\mathrm{Cov}(X_t - X_s, X_v - X_u) = 0$ where $0 \leq s \leq t \leq u \leq v$. Hence show

$$\mathrm{Var}(X_n) = \sum_{k=1}^{n} \mathrm{Var}(X_k - X_{k-1}).$$

4. An agent starts with initial capital $X_0 = 1$. At the beginning of day n he observes a new non-negative random variable Y_n, and then chooses C_n, the amount of his remaining capital to consume that day. The $(Y_n)_{1\leq n\leq N}$ are independent copies of Y with $\mathbb{E}(Y) = m$. His objective is to maximise

$$\mathbb{E}\left[\sum_{n=1}^{N} Y_n \log C_n\right]$$

subject to the constraint $\sum_{n=1}^{N} C_n = 1$. Find his optimal consumption policy. Hint: Let $X_n = 1 - \sum_{i=1}^{n-1} C_i$. Given $Y_n = y$ and $X_n = x$, look for a value function of form

$$V(n, x, y) = (y + (N - n)m)\log x + b(n, y)$$

5. Consider a market with $d = 1$ risky asset with prices $(S_n)_{n\geq 0}$ and interest rate r. Consider an investor who consumes C_n and holds θ_n shares during the interval $(n-1, n]$, where C_n and θ_n are $\mathcal{F}_{n-1}$ measurable. Given initial wealth $X_0 = x > 0$, the investor's goal is to

$$\text{maximise } \mathbb{E}\left[\sum_{k=1}^{\infty} \beta^{k-1} U(C_k)\right]$$

where $U(x) = \sqrt{x}$ is the investor's utility function and $0 < \beta < 1$ is the investor's subjective rate of discounting. Assume $S_n = S_{n-1}\xi_n$ where $(\xi_n)_{n\geq 1}$ are independent copies of the positive random variable ξ, and assume the investor's wealth X_n remains positive for all time n.

Let $\alpha = \max_t \mathbb{E}[U(1 + r + t[\xi - (1 + r)])]$ and let t^* be the maximiser. Assuming that $\alpha\beta < 1$, show that the optimal policy for the investor is to consume $C_n = (1 - \alpha^2\beta^2)X_{n-1}$ and to hold $\theta_n = \alpha^2\beta^2 t^* X_{n-1}/S_{n-1}$ shares.

6. You play the following game. There are N envelopes, with envelope n containing ξ_n pounds, where $\xi_1, \ldots, \xi_N$ are independent copies of ξ with $\mathbb{P}(\xi \leq x) = \frac{x}{\sqrt{x^2+1}}$. The envelopes are opened one at a time. After opening envelope $n < N$, you must choose whether to end the game and keep the money in the envelope, or to continue the game and open envelope $n+1$. Assuming you are risk-neutral, show that your optimal policy is to stop the first time that $\xi_n \geq \sqrt{N - n}$.

7. Consider the arbitrage-free binomial model, and introduce an exotic European contingent claim with time-N payout $Y = g(S_0, S_1, \ldots, S_N)$. Let

$$V(n, s_0, s_1, \ldots, s_n) = \frac{1}{(1+r)^{N-n}} \mathbb{E}^{\mathbb{Q}}[Y | S_0 = s_0, S_1 = s_1, \ldots, S_n = s_n]$$

and

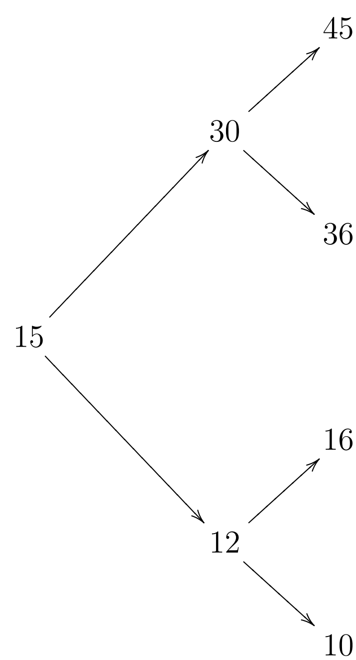
$$\theta_n = \frac{V(n, S_0, S_1, \ldots, S_{n-1}, S_{n-1}(1+b)) - V(n, S_0, S_1, \ldots, S_{n-1}, S_{n-1}(1+a))}{S_{n-1}(b-a)}$$

Show that the unique time-n no-arbitrage price of the claim is $V(n, S_0, S_1, \ldots, S_n)$ and that the claim can be replicated from intial wealth $V(0, S_0)$ by employing the trading strategy $(\theta_n)_{1 \le n \le N}$.

8. In the arbitrage-free binomial model, find the price and replication strategies for European claims with payout (a) $Y = \sqrt{S_N}$. (b) $Y = \log S_N$. (c) $Y = \sum_{k=0}^{N} S_k$.

For part (c), to what extent does your answer depend on the details of the binomial model?

9. Consider the following two-period market model with two assets. There is a riskless bank account with risk-free rate $r = 1/4$ and a stock with prices given by



Find the risk-neutral measure $\mathbb{Q}$.

Consider a European put option which strike $K = 15$ expiring at time 2. What is the no-arbitrage price of the option at time 0? What is the replicating strategy?

10. In the model of Question 9, find the time-zero price and optimal exercise policy for an American put option with strike 15 and expiry 2. Check that the value is strictly bigger than the value of the corresponding European put. Explain why, in contrast, the prices of American and Europeans calls are equal.