Stochastic Financial Models Michael Tehranchi

Example sheet 1 - Michaelmas 2020

Throughout this sheet, we consider a market with one risk-free asset with interest rate r and d risky assets with initial prices S_0 and time-1 price S_1, where $\mathbb{E}(S_1) = \mu$ and $\mathrm{Cov}(S_1) = V$. We assume that V is positive definite.

Also, we say a function is *suitable* for a problem if it is behaved well enough for the formal calculation to be justified. You may wish to discuss with your supervisor sufficient conditions for suitability.

1. Given an initial wealth $X_0 = x$, an investor tries to maximise $\mathbb{E}(X_1) - \frac{1}{2}\gamma\mathrm{Var}(X_1)$, where X_1 is her wealth at time 1, and $\gamma > 0$ is a risk-aversion parameter. Show that the optimal portfolio is mean-variance efficient and satisfies a 'mutual fund theorem'.

2. We have usually thought of the time-0 prices S_0 as given, and computed agents' optimal portfolios based on this. However, in equilibrium analysis, the prices S_0 are adjusted to clear markets. Suppose there is a total of n_i shares of asset i, and let $n = (n_1, \ldots, n_d)^\top$. Suppose that there are K agents in the market, agent k solving the maximisation problem from Problem 1 above with a risk aversion γ_k. Show that the market-clearing time-0 prices for the risky assets must be

$$S_0 = \frac{(\mu - \Gamma V n)}{1 + r}$$

where $\Gamma = (\sum_k \gamma_k^{-1})^{-1}$.

3. Consider Problem 2 above, but now suppose now that the agents decide to open a market in a contingent claim which is in zero net supply and with time-1 payout Y. Suppose the covariance matrix of the random vector $(S_1^\top, Y)^\top$ is positive definite. Find the market clearing price for the claim, and show that the initial prices S_0 of the original assets are the same as in problem 2. Show that in equilibrium none of the agents hold the contingent claim. (This is surprising, as in the real world, contingent claims are very actively traded. What are some unrealistic assumptions of this model?)

4. An agent's preferences are given by a utility function U. Suppose that the agent is indifferent between X and Y if $\mathbb{E}(X) = \mathbb{E}(Y)$ and $\mathrm{Var}(X) = \mathrm{Var}(Y)$. By considering distributions concentrated on three points, or otherwise, prove that the function U must be quadratic.

5. Suppose that the random variable Z has zero mean and that the function U is concave and increasing and suitable. Letting

$$\psi(m, t) = \mathbb{E}[U(m + tZ)],$$

show that ψ is concave: that is

$$\psi(pm_1 + qm_2, pt_1 + qt_2) \geq p\psi(m_1, t_1) + q\psi(m_2, t_2)$$

for $0 \leq p = 1 - q \leq 1$. Show that $\psi(m, t)$ increases with m and decreases with $t \geq 0$.

6. Suppose $S_1 \sim N_d(\mu, V)$, the d-dimensional Gaussian distribution. Given an initial wealth $X_0 = x$, an investor tries to maximise $\mathbb{E}[U(X_1)]$ where X_1 is her wealth at time 1. Assume that the function U is increasing and concave and suitable. Show that the optimal portfolio is mean-variance efficient.

7. Some useful facts about the Gaussian distribution for later reference. Here, Φ denotes the standard Gaussian cumulative distribution function.

(i) Suppose that $X \sim N(\mu, \sigma^2)$. Show that

(a) $\mathbb{E}[e^{\theta X} f(X)] = e^{\mu\theta + \theta^2\sigma^2/2}\mathbb{E}[f(X + \theta\sigma^2)]$ for all $\theta \in \mathbb{R}$ and suitable f,

(b) $\mathbb{E}[f(X)(X - \mu)] = \sigma^2 \mathbb{E}[f'(X)]$ for suitable f,

(c) $\mathbb{E}[\Phi(X)] = \Phi\left(\frac{\mu}{\sqrt{1+\sigma^2}}\right)$

(ii) Suppose (X, Y) have the jointly Gaussian distribution. Show that

(a) $\mathbb{E}[Y - \mathbb{E}(Y)|X] = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}[X - \mathbb{E}(X)]$

(b) $\mathrm{Cov}(f(X), Y) = \mathbb{E}[f'(X)]\mathrm{Cov}(X, Y)$ for suitable f.

8. Reconsider Problem 6. Show that the optimal portfolio of risky assets is

$$\theta^* = \frac{1}{\gamma} V^{-1}[\mu - (1 + r)S_0]$$

where

$$\gamma = -\frac{\mathbb{E}[U''(X_1^*)]}{\mathbb{E}[U'(X_1^*)]}$$

and X_1^* is the optimised time-1 wealth.

9. A contingent claim with time-1 payout Y is called attainable if there exists a scalar a and portfolio $b \in \mathbb{R}^d$ such that $Y = a + b^\top S_1$. Show that the indifference price of an attainable contingent claim does not depend on the investor's utility function U or her initial wealth $X_0 = x$.

10. Let $\pi(Y)$ the indifference price of a contingent claim with payout Y. Show that the function $t \mapsto \frac{\pi(tY)}{t}$ is decreasing.

11. Suppose an agent has utility function

$$U(x) = -e^{-\gamma x}$$

where $\gamma > 0$ is the constant coefficient of absolute risk aversion. He has initial wealth $X_0 = x$ and suppose that $S_1 \sim N_d(\mu, V)$. What is the agent's optimal portfolio? Verify that the marginal utility of optimised terminal wealth $U'(X_1^*)$ is proportional to the density of a risk-neutral probability measure $\mathbb{Q}$.

12. In the context of Problem 11, now introduce a contingent claim with payout Y where Y and S_1 are jointly normal. What is the indifference price of this claim? [Hint: first consider the case where Y and S_1 are independent. For the general case, show that there exists a portfolio $\theta \in \mathbb{R}^d$ such that $Y = \theta \cdot S_1 + Z$ where Z and S_1 are independent.] Verify that $Y \to \pi(Y)$ is concave. Verify that

$$\lim_{t \downarrow 0} \frac{1}{t}\pi(tY) = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(Y).$$